Advanced Quantitative Research Methodology, Lecture Notes: Introduction[1]

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  - Fewer proofs, more concepts, much more practical knowledge

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- Outline and all class materials at

j.mp/G2001

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4. Focus, like I will, on what you learn, not your grades: Especially when we work on papers, I will treat you like a colleague, not a student

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- When are Gary's office hours?

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- The number of new methods is increasing fast
- Most important methods originate *outside* the discipline of statistics (random assignment, experimental design, survey research, machine learning, MCMC methods, . . .). Statistics: abstracts, proves formal properties, generalizes, and distributes results back out.

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- (Valuable for the job market)
- Part of a massive change in the evidence base of the social sciences: (a) surveys, (b) end of period government stats, and (c) one-off studies of people, places, or events $\rightsquigarrow$ numerous new types and huge quantities of (big) data

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- Instead, we teach you the fundamentals, the underlying theory of inference, from which most statistical models are developed.
- This helps us separate the conventions from underlying statistical theory. (How to get an F in Econometrics: follow advice from Psychometrics. Works in reverse too, even when the foundations are identical.)

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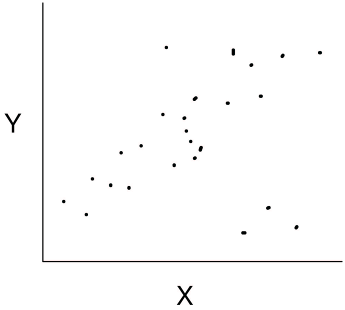

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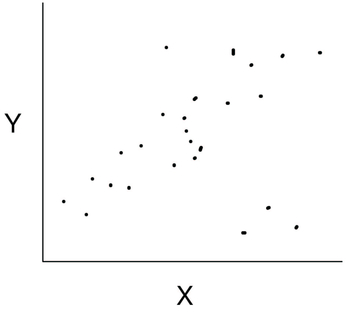

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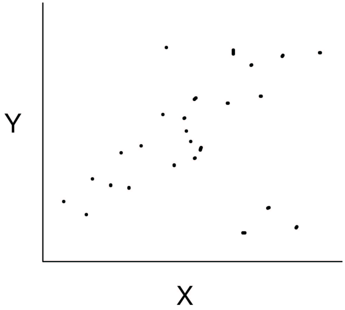

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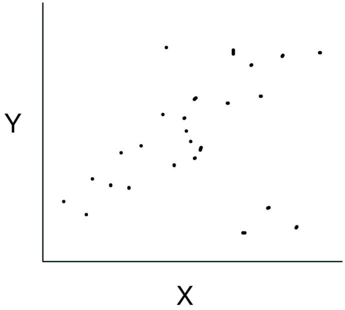

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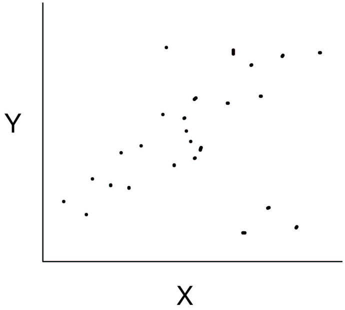

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- from a theory of inference, and for a substantive purpose (like causal estimation, prediction, etc.)

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  2. Teach the fundamentals, then do examples in great detail. Introduce math in almost as much depth, and only when needed.

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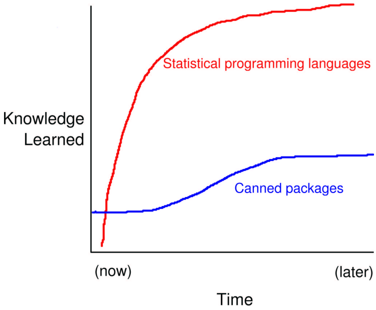

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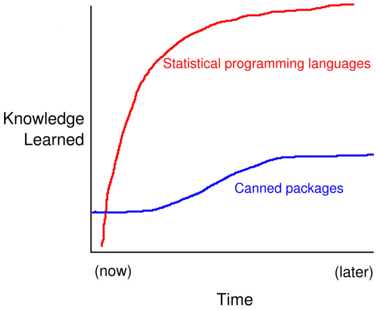

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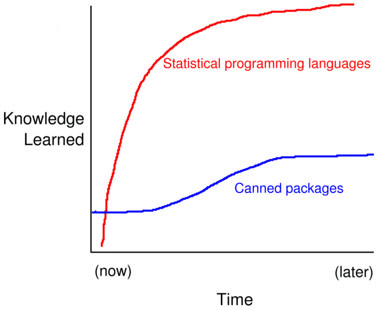

- We'll use R — a free open source program, a commons, a movement
- We'll also use an R program called Zelig (Imai, King, and Lau, 2012) which will greatly simplify R and help you up the steep slope fast (see j.mp/Zelig4)

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- Models of dirt on airplanes, vs models of aerodynamics

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Inference (using facts you know to learn facts you don't know) v summarization

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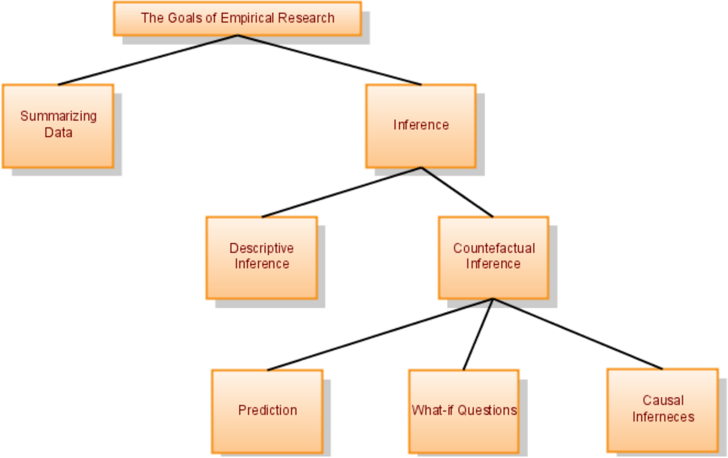

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- Dependent (or "outcome") variable: Y is $n \times 1$.

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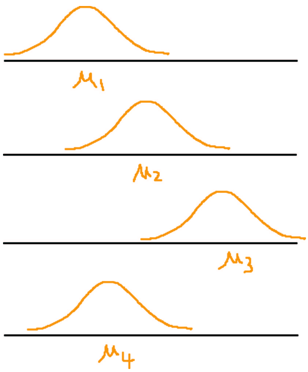

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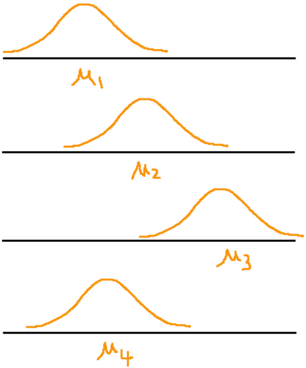

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- Fundamental uncertainty: Represented by the stochastic component. Exists no matter what the researcher does; no matter how large n is.

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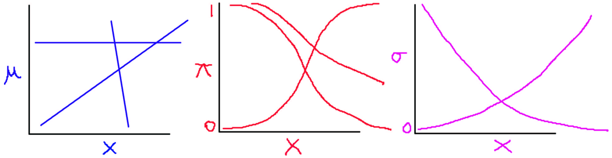

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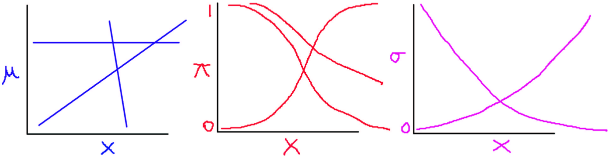

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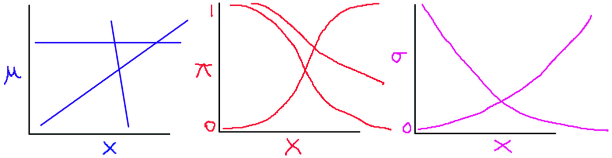

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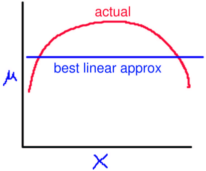

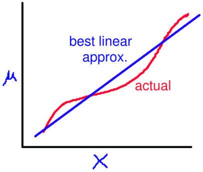

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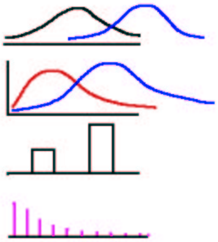

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  - Later: relax functional form and distributional assumptions, or preprocess data (via matching, etc.) to avoid their consequences

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- $\rightsquigarrow$ Experiments: students get the right answer far more frequently by using simulation than math

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4. Example: estimate the mean of the population	4. Example: Approximate the mean of the distribution

Simulation examples for solving probability problems

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```
Four runs: .538, .550, .547, .524
```

Let's Make a Deal

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In Let's Make a Deal, Monte Hall offers what is behind one of three doors. Behind a random door is a car; behind the other two are goats. You choose one door at random. Monte peeks behind the other two doors and opens the one (or one of the two) with the goat. He asks whether you'd like to switch your door with the other door that hasn't been opened yet. Should you switch?

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for (i in 1:sims) {
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  choice <- sample(doors, 1)
  if (WinDoor == choice)                      # no switch
  WinNoSwitch <- WinNoSwitch + 1
  doorsLeft <- doors[doors != choice]         # switch
  if (any(doorsLeft == WinDoor))
  WinSwitch <- WinSwitch + 1
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}
cat("Prob(Car | no switch)=", WinNoSwitch/sims, "\n")
cat("Prob(Car | switch)=", WinSwitch/sims, "\n")
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Let's Make a Deal

Pr(car|No Switch)	Pr(car|Switch)
.324	.676
.345	.655
.320	.680
.327	.673

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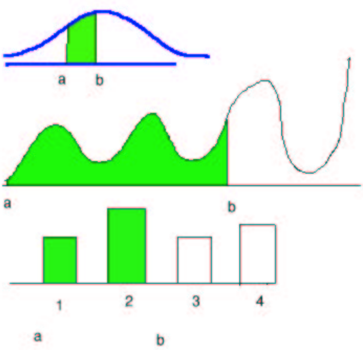
2. $\mathrm{P}(Y) \geq 0$ for every Y

Computing Probabilities from Densities



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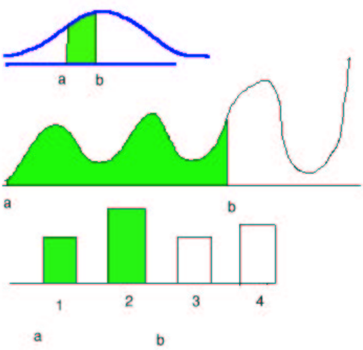

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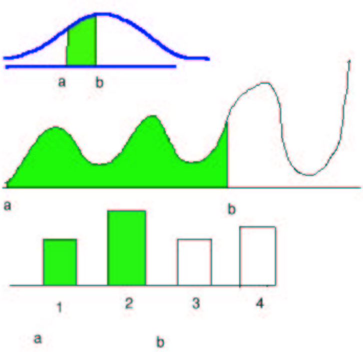

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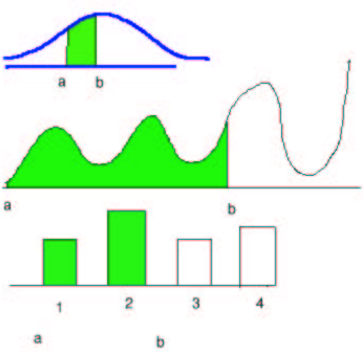

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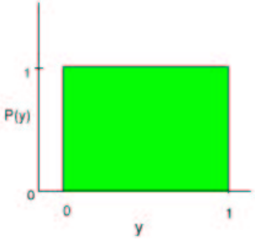

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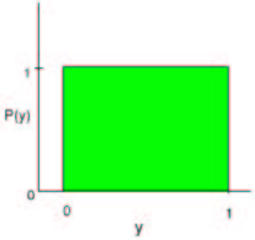

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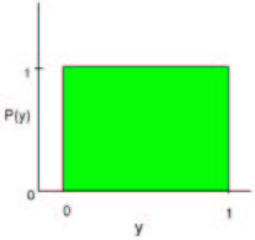

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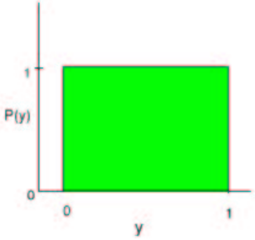

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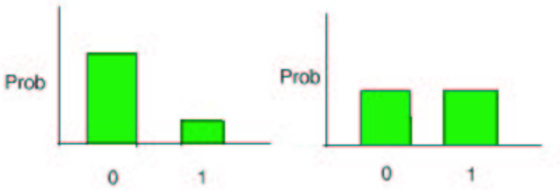
  - $\implies \Pr(Y_i = y|\pi_i) = \pi_i^y(1 - \pi_i)^{1-y}$
  - Alternative notation: $\Pr(Y_i = y|\pi_i) = \text{Bernoulli}(y|\pi_i) = f_b(y|\pi_i)$

Graphical summary of the Bernoulli



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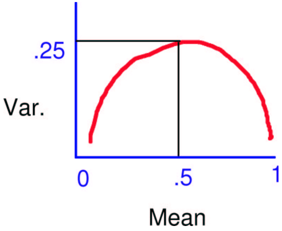

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sims <- 1000     # set parameters
bernpi <- 0.2
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"Discretization" for random draws from discrete pmfs, given uniform random numbers

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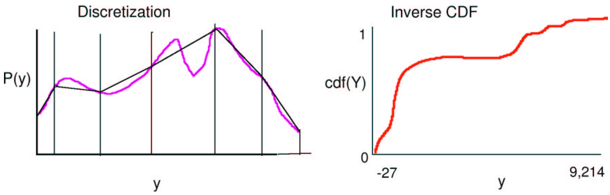

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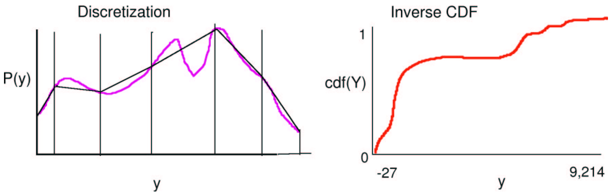

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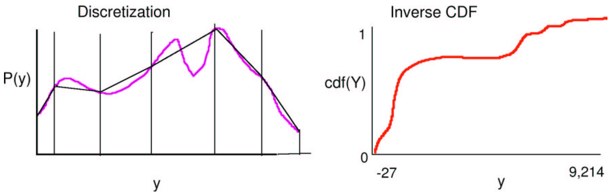

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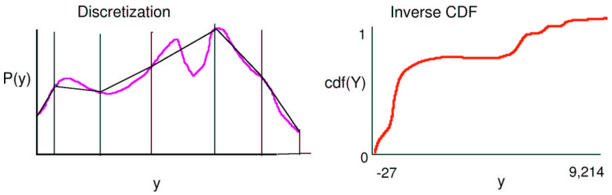

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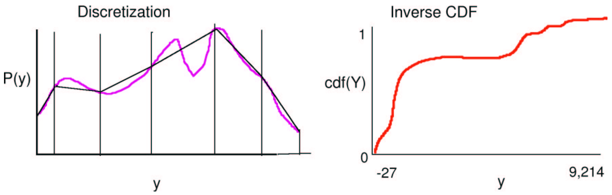

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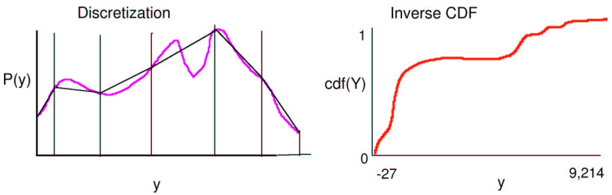
Inverse CDF method for random draws from continuous pdfs given uniform random numbers



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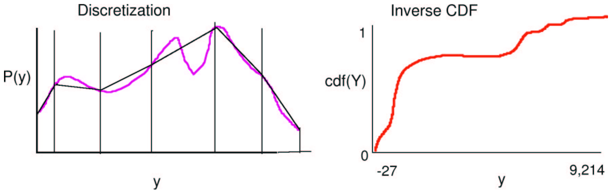

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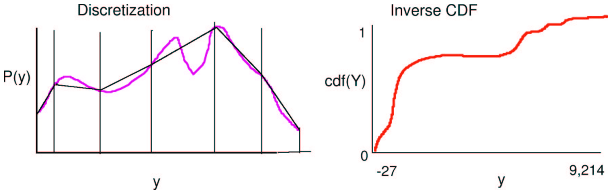

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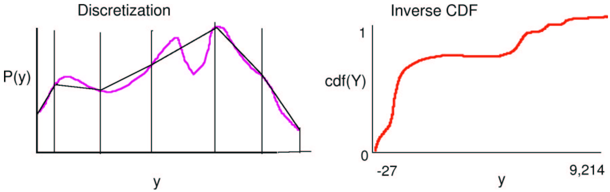

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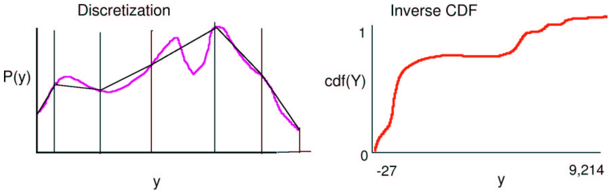

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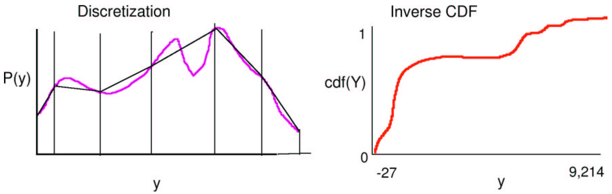
- Then $F^{-1}(U)$ gives a random draw from $f(Y)$.

A Refined Discretization method



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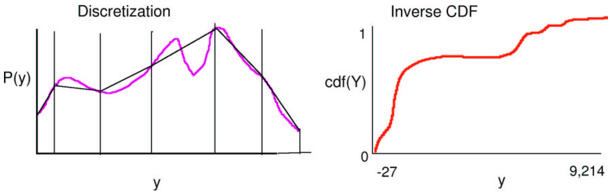

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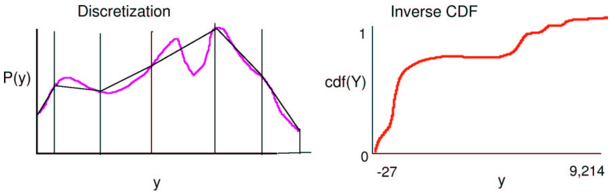

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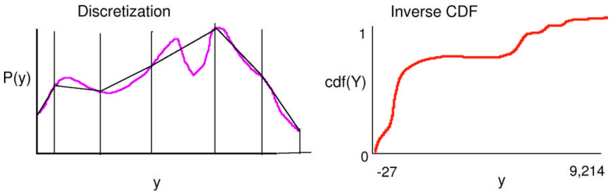

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- Drawing random numbers from arbitrary multivariate densities: now an enormous literature

Stop here

We will stop here this year and skip to the next set of slides. Please refer to the notes below for further information on probability densities and random number generation.

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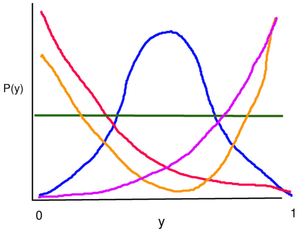

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Reparameterization like this will be key throughout the course.

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- Add up the $\tilde{z}$'s to get $y = \sum_j^N \tilde{z}_j$, which is a draw from the beta-binomial.

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- Pr(event at time t | all events up to time $t-1$) is constant for all t.

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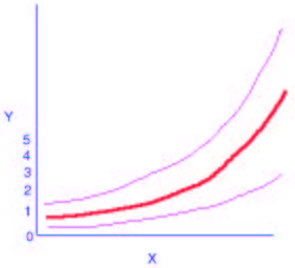

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- How to simulate? We'll use canned random number generators.

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$$\text{gamma}(y|\phi, \sigma^2) = \frac{y^{\phi(\sigma^2-1)^{-1}-1} e^{-y(\sigma^2-1)^{-1}}}{\Gamma[\phi(\sigma^2-1)^{-1}](\sigma^2-1)^{\phi(\sigma^2-1)^{-1}}}$$

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&= \frac{\Gamma\left(\frac{\phi}{\sigma^2-1}+y_i\right)}{y_i!\Gamma\left(\frac{\phi}{\sigma^2-1}\right)}\left(\frac{\sigma^2-1}{\sigma^2}\right)^{y_i}\left(\sigma^2\right)^{\frac{-\phi}{\sigma^2-1}}
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- Simulating *once* from this density produces k numbers. Special algorithms are used to generate normal random variates (in R, `mvrnorm()`, from the `MASS` library).

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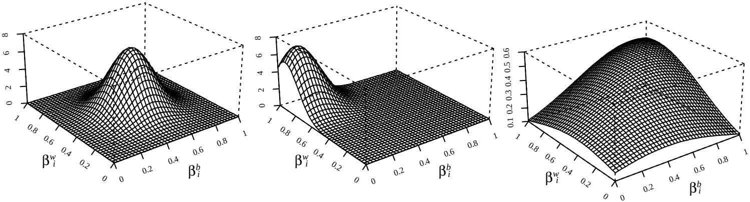

(a) 0.5 0.5 0.15 0.15 0 (b) 0.1 0.9 0.15 0.15 0 (c) 0.8 0.8 0.6 0.6 0.5

Parameters are μ_1, μ_2, σ_1, σ_2, and ρ.

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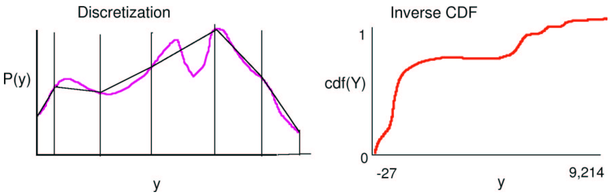

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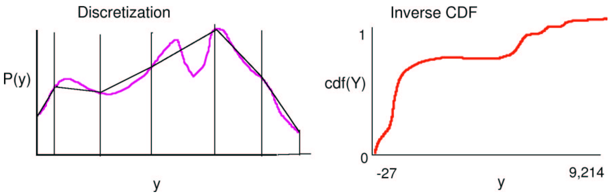

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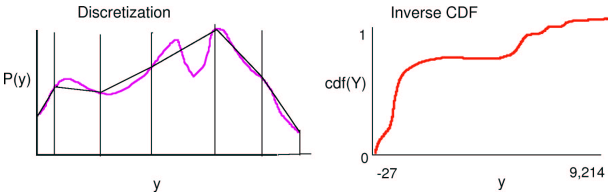

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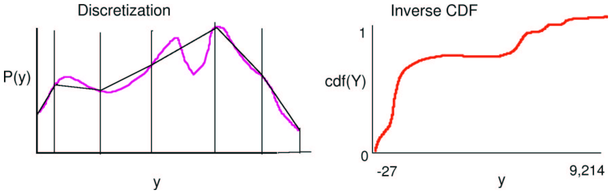

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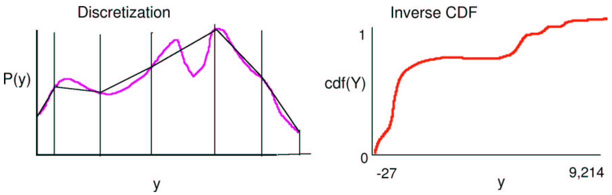

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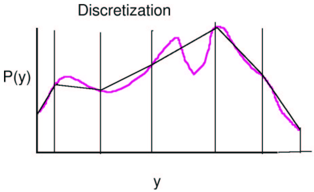
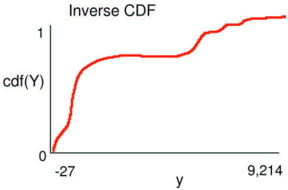
- Not feasible for multivariate random number generation

Inverse CDF method for random draws from continuous pdfs given uniform random numbers





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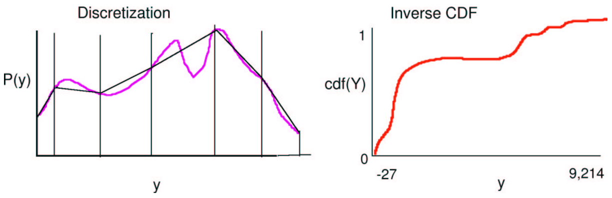

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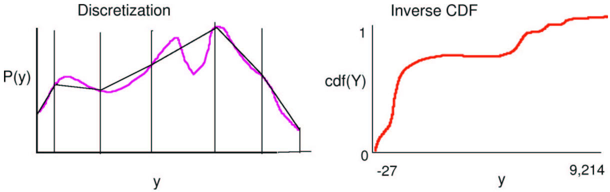

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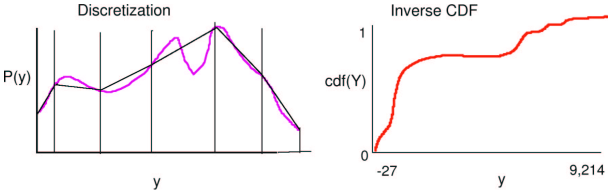

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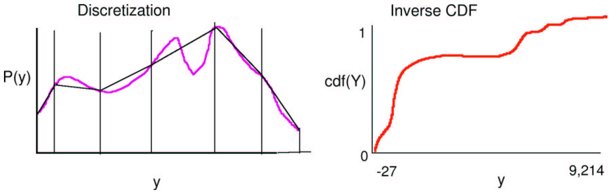

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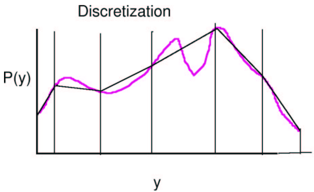
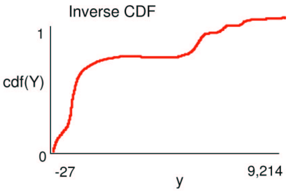
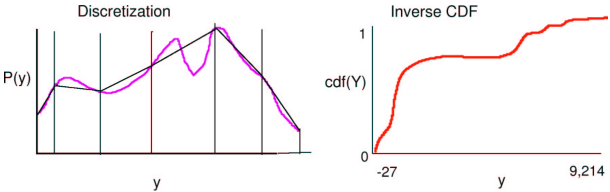
- Then $F^{-1}(U)$ gives a random draw from $f(Y)$.

A Refined Discretization method



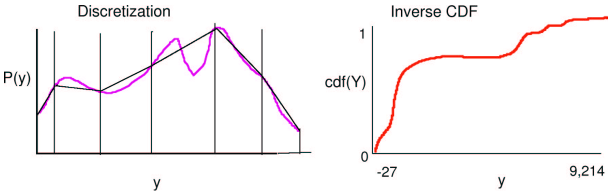


A Refined Discretization method



- Choose interval randomly as above

A Refined Discretization method



- Choose interval randomly as above
- Draw a number within an interval by the inverse CDF method applied to the trapezoidal approximation.