

Advanced Quantitative Research Methodology, Lecture Notes: **Single Equation Models**¹

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Binary Variable Regression Models

The logistic regression (or “logit”) model:

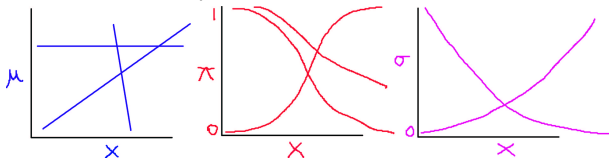
① Stochastic component:

$$Y_i \sim Y_{\text{Bern}}(y_i|\pi_i) = \pi_i^{y_i}(1 - \pi_i)^{1-y_i} = \begin{cases} \pi_i & \text{for } y = 1 \\ 1 - \pi_i & \text{for } y = 0 \end{cases}$$

② Systematic Component:

$$\Pr(Y_i = 1|\beta) \equiv E(Y_i) \equiv \pi_i = \frac{1}{1 + e^{-x_i\beta}}$$

③ Y_i and Y_j are independent $\forall i \neq j$, conditional on X
(The graph in the middle:)



Binary Variable Regression Models

The probability density of all the data:

$$P(y|\pi) = \prod_{i=1}^n \pi_i^{y_i} (1 - \pi_i)^{1-y_i}$$

The log-likelihood:

$$\begin{aligned} \ln L(\pi|y) &= \sum_{i=1}^n \{y_i \ln \pi_i + (1 - y_i) \ln(1 - \pi_i)\} \\ &= \sum_{i=1}^n \left\{ -y_i \ln \left(1 + e^{-x_i \beta} \right) + (1 - y_i) \ln \left(1 - \frac{1}{1 + e^{-x_i \beta}} \right) \right\} \\ &= - \sum_{i=1}^n \ln \left(1 + e^{(1-2y_i)x_i \beta} \right). \end{aligned}$$

What do we do with this?

Interpreting Functional Forms

Running Example is logit:

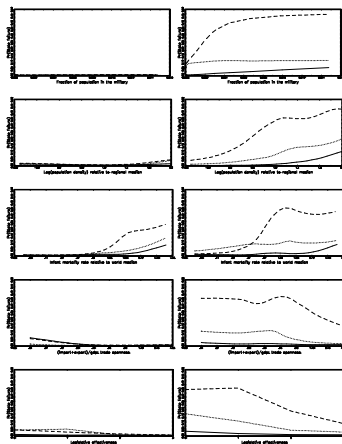
$$\pi_i = \frac{1}{1 + e^{-x_i\beta}}$$

Methods:

1. Graphs.

- (a) Can use desired instead of observed X 's
- (b) Can try entire surface plot for a small number of X 's
- (c) Marginal effects: Can hold "other variables" constant at their means, a typical value, or at their observed values
- (d) Average effects: compute effects for every observation and average

Interpreting Functional Forms



Example Marginal Effect Plot of a neural network model (about which more later). Full democracies (dotted), partial democracies (dashed), and autocracies (solid). From Gary King and Langche Zeng. "Improving Forecasts of State Failure," *World Politics*, Vol. 53, No. 4 (July, 2001): 623-58.

Interpreting Functional Forms

2. **Fitted Values** for selected combinations of X 's, or “typical” people or types:

Sex	Age	Home	Income	Pr(vote)
Male	20	Chicago	\$33,000	0.20
Female	27	New York City	\$43,000	0.28
Male	50	Madison, WI	\$55,000	0.72
⋮				

For any quantity but a probability, always also include a measure of uncertainty (standard error, confidence interval, etc.)

Interpreting Functional Forms

3. First Differences (called Risk Differences in epidemiology)

- (a) Define X_s (starting point) and X_e (ending point) as $k \times 1$ vectors of values of X . Usually all values are the same but one.
- (b) First difference = $g(X_e, \hat{\beta}) - g(X_s, \hat{\beta})$
- (c) $D = \frac{1}{1+e^{-X_e\hat{\beta}}} - \frac{1}{1+e^{-X_s\hat{\beta}}}$
- (d) Better (and necessary to compute se's): do by simulation (we'll repeat the details soon)

Variable	From		To	First Difference
Sex	Male	→	Female	.05
Age	65	→	75	-.10
Home	NYC	→	Madison, WI	.26
Income	\$35,000	→	\$75,000	.14

4. Derivatives (some great rules of thumb)

$$\frac{\partial \pi_i}{\partial X_j} = \frac{\partial \frac{1}{1+e^{-X\beta}}}{\partial X_j} = \hat{\beta}_j \hat{\pi}_i (1 - \hat{\pi}_i)$$

(a) Max value of logit derivative: $\hat{\beta} \times 0.5(1 - 0.5) = \hat{\beta}/4$

(b) Max value for probit [$\pi_i = \Phi(X_i\beta)$] derivative: $\hat{\beta} \times 0.4$

Same Logit Model, Different Justification and Interpretation

Let Y^* be a continuous unobserved variable. Health, propensity to vote, etc.

A model:

$$Y_i^* \sim P(y_i^* | \mu_i) \\ \mu_i = x_i \beta$$

with observation mechanism:

$$y_i = \begin{cases} 1 & y^* \leq \tau \text{ if } i \text{ is alive} \\ 0 & y^* > \tau \text{ if } i \text{ is dead} \end{cases}$$

Since Y^* is unobserved anyway, define the threshold as $\tau = 0$. (Plus the same independence assumption, which from now on is assumed implicit.)

Same Logit Model, Different Justification and Interpretation

1. If Y^* is observed and $P(\cdot)$ is normal, this is a regression.
2. If only y_i is observed, and Y^* is standardized logistic (which looks close to the normal),

$$P(y_i^*|\mu_i) = \text{STL}(y_i^*|\mu_i) = \frac{\exp(y_i^* - \mu_i)}{[1 + \exp(y_i^* - \mu_i)]^2}$$

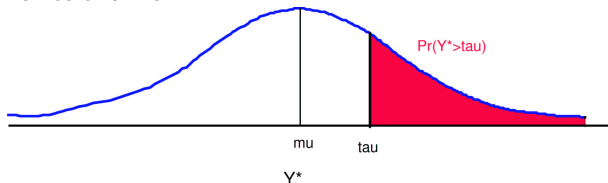
then we get a logit model.

Same Logit Model, Different Justification and Interpretation

3. The derivation:

$$\begin{aligned}\Pr(Y_i = 1|\mu_i) &= \Pr(Y_i^* \leq 0) \\ &= \int_{-\infty}^0 \text{STL}(y_i^*|\mu_i) dy_i^* \\ &= F_{\text{stl}}(0|\mu_i) \quad [\text{the CDF of the STL}] \\ &= [1 + \exp(-X_i\beta)]^{-1}\end{aligned}$$

The same functional form!



Same Logit Model, Different Justification and Interpretation

4. For the **Probit Model**, we modify:

$$P(y_i^*|\mu_i) = N(y_i^*|\mu_i, 1)$$

implying

$$\Pr(Y_i = 1|\mu) = \int_{-\infty}^0 N(y_i^*|\mu_i, 1)dy_i^* = \Phi(X_i\beta)$$

5. $\implies$ interpret β as regression coefficients of Y^* on X : $\hat{\beta}_1$ is what happens to Y^* on average (or μ_i) when X_1 goes up by one unit, holding constant the other explanatory variables (and conditional on the model). In probit, one unit of Y^* is one standard deviation.

An Econometric Interpretation: Utility Maximization

1. Let U_i^D be the utility for the Democratic candidate; and U_i^R be the utility for the Republican candidate.
2. Assume U_i^D and U_i^R are independent
3. Assume $U_i^k \sim P(U_i^k | \eta_i^k)$ for $k = \{D, R\}$.
4. Let $Y^* \equiv U_i^D - U_i^R$ and apply the same interpretation as above: If $y^* > 0$, choose the Democrat, otherwise, choose the Republican.
5. If $P(\cdot)$ is normal, we get a Probit model
6. If $P(\cdot)$ is generalized extreme value, we get logit.
 - Of the three justifications for the same binary model, which do you prefer?
 - Which would enable you to choose logit over probit?

How Not to Present Statistical Results

TABLE 1	
Predicting Which Ethnic Group Conquered Most of Bosnia	
Attention to Bosnia crisis	.609**
Age	.007**
Education	.289**
Family income	.151**
Race (non-White/White)	.695**
Gender (female/male)	.789**
Region (South/non-South)	.076
Network coverage	.000
Education $\times$ Time	-.003*
Time in months	.078**
Constant	-9.257**
Number	7,021
-2 log-likelihood	7,215.231
Goodness of fit	6,789.45
Cox & Snell R^2	.212
Nagelkerke R^2	.295
Overall correct classification (%)	73.96

SOURCE: *Times Mirror* polls from September 1992, January 1993, September 1993, January 1994, and June 1995.

NOTE: Unstandardized coefficients for logistic regression. Dependent variable is knowledge of which group conquered most of Bosnia.

* $p \leq .05$, two-tailed. ** $p \leq .01$, two-tailed.

1. This one is typical of current practice, not that unusual.
2. What do these numbers mean?
3. Why so much whitespace? Can you connect cols A and B?

How Not to Present Statistical Results: Continued

TABLE 1
Predicting Which Ethnic Group Conquered Most of Bosnia

Attention to Bosnia crisis	.609**
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* $p \leq .05$, two-tailed. ** $p \leq .01$, two-tailed.

4. What does the star-gazing add?
5. Can any be interpreted as causal estimates?
6. Can you compute a quantity of interest from these numbers?

Interpretation and Presentation

1. Statistical presentations should
 - (a) Convey numerically precise estimates of the quantities of substantive interest,
 - (b) Include reasonable measures of uncertainty about those estimates,
 - (c) Require little specialized knowledge to understand.
 - (d) Include no superfluous information, no long lists of coefficients no one understands, no star gazing, etc.
2. For example: Other things being equal, an additional year of education would increase your annual income by \$1,500 on average, plus or minus about \$500.
3. Your work should satisfy someone like me and someone like my grandmother.
4. Reading assignment: (at my web page) the handout on papers, and King, Tomz, Wittenberg, "Making the Most of Statistical Analyses: Improving Interpretation and Presentation" American Journal of Political Science, Vol. 44, No. 2 (March, 2000): 341-355.

For any statistical model

$$Y_i \sim f(y_i | \theta_i, \alpha)$$

stochastic

$$\theta_i = g(x_i, \beta)$$

systematic

recall that we have

1. *Estimation uncertainty*: Lack of knowledge of β and α . Vanishes as n gets larger.
2. *Fundamental uncertainty*: Represented by the stochastic component. Exists no matter what the researcher does.

Simulating Estimation Uncertainty: By Simulating the Parameters

To take one random draw of all the parameters $\gamma = \text{vec}(\beta, \alpha)$ from their “sampling distribution” (or “posterior distribution” with a flat prior):

1. Estimate the model by maximizing the likelihood function, record the point estimates $\hat{\gamma}$ and variance matrix $\hat{V}(\hat{\gamma})$.
2. Draw the vector γ from the multivariate normal distribution:

$$\gamma \sim N(\hat{\gamma}, \hat{V}(\hat{\gamma}))$$

Denote the draw $\tilde{\gamma} = \text{vec}(\tilde{\beta}, \tilde{\alpha})$, which has k elements.

Simulating the Distribution of Predicted Values

Predicted values can be for:

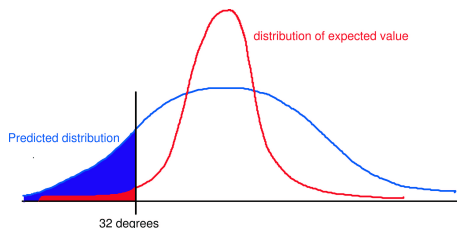
1. Forecasts: about the future
2. Farcasts: about some area for which you have no y
3. Nowcasts: about the current data (perhaps to reproduce it to see whether it fits)

To simulate *one* predicted value, follow these steps:

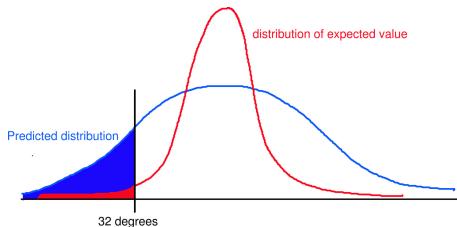
1. Draw one value of $\tilde{\gamma} = \text{vec}(\tilde{\beta}, \tilde{\alpha})$.
2. Choose a predicted value to compute, defined by one value for each explanatory variable as the vector X_c .
3. Extract simulated $\tilde{\beta}$ from $\tilde{\gamma}$; compute $\tilde{\theta}_c = g(X_c, \tilde{\beta})$ (from systematic component)
4. Simulate outcome variable $\tilde{Y}_c \sim f(\tilde{\theta}_c, \tilde{\alpha})$ (from stochastic component)

Repeat algorithm say $M = 1000$ times, to produce 1000 predicted values. Use these to compute a histogram for the full posterior, the average, variance, percentile values, or others.

How the Distribution of Expected and Predicted Values Differ



1. Predicted values: draws of Y that are or could be observed
2. Expected values: draws of fixed features of the distribution of Y , such as $E(Y)$.
3. Predicted values: include estimation and fundamental uncertainty.
4. Expected values: average away fundamental uncertainty



5. The variance of expected values (but not predicted values) go to 0 and n gets large.
6. **Example use of predicted value distribution:** probability of temperature colder than 32° tomorrow. (Predicted temperature is uncertain because we have to estimate it *and* because of natural fluctuations.)
7. **Example use of expected value distribution:** probability the average temperature on days like tomorrow will be colder than 32° . (Expected temperature is only uncertain because we have to estimate it; natural fluctuations in temperature doesn't affect the average.)
8. Which to use for causal effects & first differences?

Simulating the Distribution of Expected Values: An Algorithm

1. Draw **one** value of $\tilde{\gamma} = \text{vec}(\tilde{\beta}, \tilde{\alpha})$.
2. Choose **one** value for each explanatory variable (X_c is a vector)
3. Taking the **one** set of simulated $\tilde{\beta}$ from $\tilde{\gamma}$, compute $\tilde{\theta}_c = g(X_c, \tilde{\beta})$ (from the systematic component)
4. Draw **m values** of the outcome variable $\tilde{Y}_c^{(k)}$ ($k = 1, \dots, m$) from the stochastic component $f(\tilde{\theta}_c, \tilde{\alpha})$. (This step simulates fundamental uncertainty.)
5. Average over the fundamental uncertainty by calculating the mean of the **m simulations** to yield **one** simulated expected value
$$\tilde{E}(Y_c) = \sum_{k=1}^m \tilde{Y}_c^{(k)} / m.$$

Simulating Expected Values: Notes

1. When $m = 1$, this algorithm produces predicted values.
2. With large m , this algorithm better represents and averages over the fundamental uncertainty.
3. Repeat M times (say 1000), with results differing only due to estimation uncertainty
4. Use to compute a histogram, average, standard error, confidence interval, etc.
5. When $E(Y_c) = \theta_c$, we can skip the last two steps. E.g., in the logit model, once we simulate π_i , we don't need to draw Y and then average to get back to π_i . (If you're unsure, do it anyway!)

Simulating First Differences

To draw one simulated first difference:

1. Choose vectors X_s , the starting point, X_e , the ending point.
2. Apply the expected value algorithm twice, once for X_s and X_e (but reuse the random draws).
3. Take the difference in the two expected values.
4. (To save computation time, and improve approximation, use the same simulated β in each.)

Tricks for Simulating Parameters

1. Simulate all parameters (in γ), including ancillary parameters, together, unless you know they are orthogonal.
2. Reparameterize to unbounded scale to
 - make $\hat{\gamma}$ converge more quickly in n (and so work better with small n) to a multivariate normal. (MLEs don't change, but the posteriors do.)
 - make the maximization algorithm work faster without constraints
3. To do this, all estimated parameters should be unbounded and logically symmetric. E.g.,
 - $\sigma^2 = e^\eta$ (i.e., wherever you see σ^2 , in your log-likelihood function, replace it with e^η)
 - For a probability, $\pi = [1 + e^{-\eta}]^{-1}$ (a logit transformation).
 - For $-1 \leq \rho \leq 1$, use $\rho = (e^{2\eta} - 1)/(e^{2\eta} + 1)$ (Fisher's Z transformation)

In all 3 cases, η is unbounded: estimate it, simulate from it, and reparameterize back to the scale you care about.

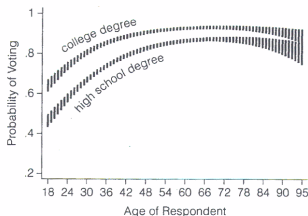
Tricks for Simulating Quantities of Interest

1. Unless you're sure, always compute simulations of Y and use that as a basis for creating simulations of other quantities. (This will get all information from the model in the simulations.)
2. Simulating functions of Y
 - (a) If some function of Y , such as $\ln(Y)$, is used, simulate $\ln(Y)$ and then apply the inverse function $\exp(\ln(Y))$ to reveal Y .
 - (b) The usual, but wrong way: Regress $\ln(Y)$ on X , compute predicted value $\widehat{\ln(Y)}$ and exponentiate.
 - (c) Its wrong because the regression estimates $E[\ln(Y)]$, but $E[\ln(Y)] \neq \ln[E(Y)]$, so $\exp(E[\ln(Y)]) \neq Y$
 - (d) More generally, $E(g[Y]) \neq g[E(Y)]$, unless $g[\cdot]$ is linear.
3. Check the *approximation error* of your simulation algorithm: Run it twice, check the number of digits of precision that don't change. If its not enough for your tables, increase M (or m) and try again.
4. Analytical calculations and other tricks can speed simulation, or precision.
5. **Clarify** does this all automatically in Stata. **Zelig** does the same and more in R.

Replication of Rosenstone and Hansen from King, Tomz and Wittenberg (2000)

1. Logit of reported turnout on Age, Age², Education, Income, and Race
2. Quantity of Interest: (nonlinear) effect of age on $\Pr(\text{vote}|X)$, holding constant Income and Race.
3. Use $M = 1000$ and compute 99% CI:

FIGURE 1 Probability of Voting by Age

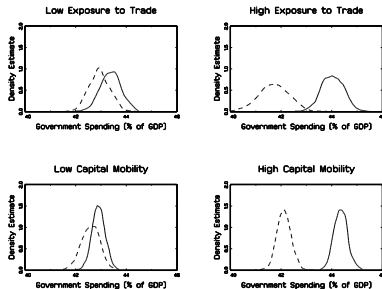


Vertical bars indicate 99-percent confidence intervals

To create this graph, simulate:

1. Set age=24, education=high school, income=average, Race=white
2. Run logistic regression
3. Simulate 1000 $\tilde{\beta}$'s
4. Compute 1000 $\tilde{\pi}_i = [1 + e^{x_i \tilde{\beta}}]^{-1}$
5. Sort in numerical order
6. Take 5th and 995th values as the 99% confidence interval
7. Plot a vertical line on the graph at age=24 representing the CI.
8. Repeat for other ages and for college degree.

Replication of Garrett (King, Tomz and Wittenberg 2000)



- Dependent variable: Government Spending as % of GDP
- Key explanatory variable: left-labor power (high = solid line; low = dashed)
- Garrett used only point estimates to distinguish the eight quantities represented above. What new information do we learn with this approach?
- Left-labor power only has a clear effect when exposure to trade or capital mobility is high.
- How could we summarize this information with less real estate?

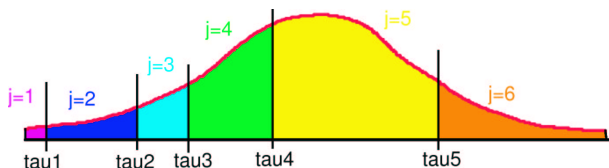
Ordered Dependent Variable Models

The model

$$Y_i^* \sim \text{STN}(y_i^* | \mu_i)$$
$$\mu_i = x_i \beta$$

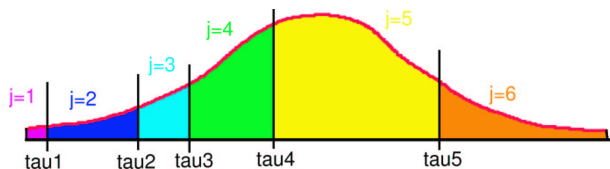
Observation mechanism

$$y_{ij} = \begin{cases} 1 & \text{if } \tau_{j-1,i} \leq y_i^* \leq \tau_{j,i} \\ 0 & \text{otherwise} \end{cases}$$



Ordinal Dependent Variable Models: Notes

1. If we switch to $Y_i^* \sim \text{STL}(y_i^* | \mu_i)$, we get an ordinal logit model
2. One dichotomous variable Y_{ji} for each category j , only one of which is 1; the others are 0.
3. If Y_i^* is observed, this is a linear-normal regression model
4. If a dichotomous realization of Y^* is observed, its a probit model
5. This is the same model, and the same parameters are being estimated; only the observation mechanism differs.



Deriving the likelihood function

First the probability of each observation, then the joint probability.

$$\begin{aligned}\Pr(Y_{ji} = 1) &= \Pr(\tau_{j-1} \leq Y_i^* \leq \tau_j) \\ &= \int_{\tau_{j-1}}^{\tau_j} \text{STN}(y_i^* | \mu_i) dy_i^* \\ &= F_{\text{stn}}(\tau_j | \mu_i) - F_{\text{stn}}(\tau_{j-1} | \mu_i) \\ &= F_{\text{stn}}(\tau_j | \mathbf{x}_i \beta) - F_{\text{stn}}(\tau_{j-1} | \mathbf{x}_i \beta)\end{aligned}$$

The joint probability is then:

$$P(Y) = \prod_{i=1}^n \left[\prod_{j=1}^J \Pr(Y_{ji} = 1)^{y_{ij}} \right]$$

Bracketed portion has only one factor for each i .

Deriving the likelihood function

The Log-likelihood:

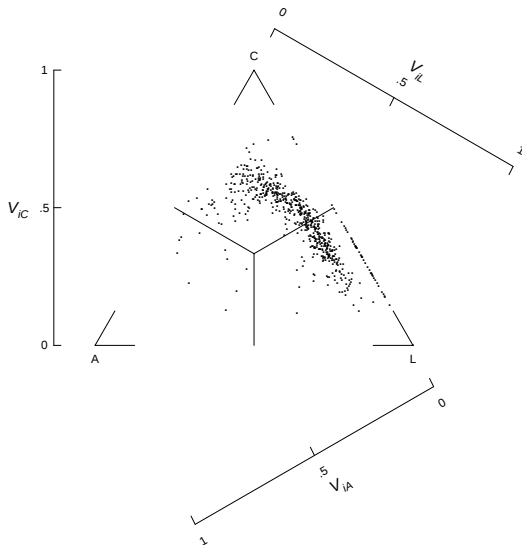
$$\begin{aligned}\ln L(\beta, \tau|y) &= \sum_{i=1}^n \sum_{j=1}^J y_{ij} \ln \Pr(Y_{ji} = 1) \\ &= \sum_{i=1}^n \sum_{j=1}^J y_{ij} \ln [F_{stn}(\tau_j|x_i\beta) - F_{stn}(\tau_{j-1}|x_i\beta)]\end{aligned}$$

(Careful of constraints during optimization: $\tau_{j-1} < \tau_j, \forall j$)

Interpretation: Ordinal Probit

1. Coefficients are the linear effect of X on Y^* in standard deviation units
2. Predictions from the model are J probabilities that sum to 1.
3. One first difference has an effect on all J probabilities.
4. When one probability goes up, at least one of the others must go down.
5. Can use ternary diagrams if $J = 3$

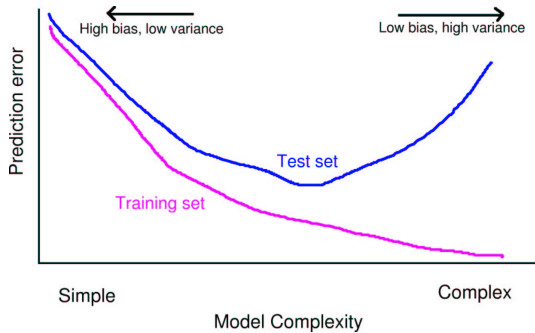
Representing 3 variables, with $Y_j \in [0, 1]$ and $\sum_{j=1}^3 Y_j = 1$



How do you know which model is better?

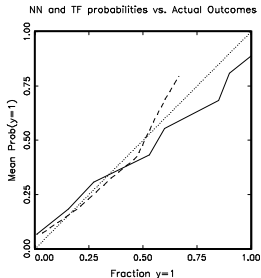
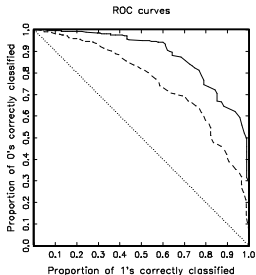
1. Out-of-sample forecasts (or farcasts)

- (a) Your job: find the underlying (persistent) structure, not the idiosyncratic features of any one data set.
- (b) Set aside some (test) data.
- (c) Fit your model to the rest (the training data).
- (d) Make predictions with training set; compare to the test set.
- (e) Comparisons to average prediction and full distribution.
- (f) E.g., if a set of predictions have $\Pr(y = 1) = 0.2$, then 20% of these observations in the test set should be 1s.
- (g) The best test sets are really out of sample, not even available yet.
- (h) If the world changes, an otherwise good model will fail. But it's still the right test.

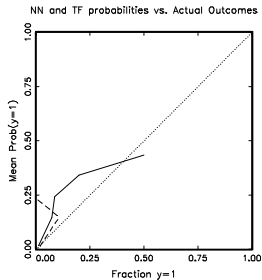
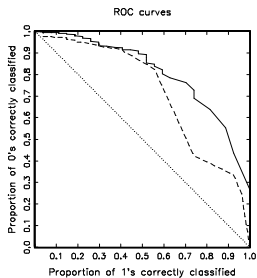


(See Trevor Hastie et al. 2001. *The Elements of Statistical Learning*, Springer, Chapter 7: Fig 7.1.)

- (i) Binary variable predictions require a normative decision.
- i. Let C be number of times more costly misclassifying a 1 is than a 0.
 - ii. C must be chosen independently of the data.
 - iii. C could come from your philosophical justification, survey of policy makers, a review of the literature, etc.
 - iv. People often choose $C = 1$, but without justification.
 - v. Decision theory: choose $Y = 1$ when $\hat{\pi} > 1/(1 + C)$ and 0 otherwise.
 - If $C = 1$, predict $y = 1$ when $\hat{\pi} > 0.5$
 - If $C = 2$, predict $y = 1$ when $\hat{\pi} > 1/3$
 - vi. Only with C chosen can we compute (a) % of 1s correctly predicted and (b) % of 0s correctly predicted, and (c) patterns in errors in different subsets of the data or forecasts.
- (j) If you can't justify a choice for C , use ROC (receiver-operator characteristic) curves
- (i) Compute %1s and %0s correctly predicted for every possible value of C
 - (ii) Plot %1s by %0s
 - (iii) Overlay curves for several models on the same graph.
 - (iv) If one curve is above another the whole way, then that model *dominates* the other. It's better no matter your normative decision (about C)
 - (v) Otherwise, one model is better than the other in only given specified ranges of C (i.e., for only some normative perspectives).



In sample ROC, on left (from Gary King and Langche Zeng. “Improving Forecasts of State Failure,” *World Politics*, Vol. 53, No. 4 (July, 2001): 623-58)



Out of sample ROC on left.

2. Cross-validation

- (a) The idea: set aside k observations as the “test set”: evaluate; and then set aside another set of k observations. Repeat multiple times; report performance averaged over subsets
- (b) Useful for smaller data sets; real test sets are better.

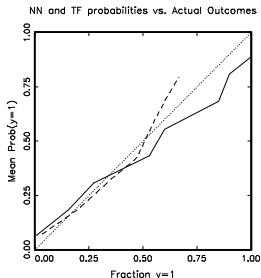
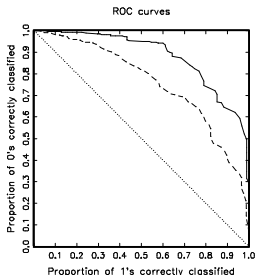
3. Fit, in general: Look for all possible observable implications of a model, and compare to observations. (Think. Be creative here!)

4. Fit: continuous variables

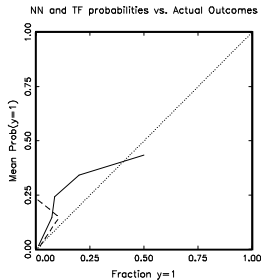
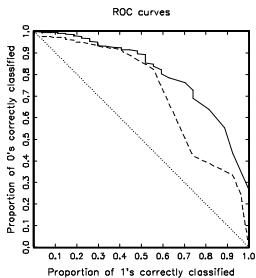
- (a) The usual regression diagnostics
- (b) E.G., plots of $e = y - \hat{y}$ by X , Y or $\hat{y}$
- (c) Check more than the means. E.g., plot e by $\hat{y}$ and draw a line at 0 and at $\pm 1, 2$ se's. 66%, 95% of the observations should fall between the lines.
- (d) For graphics:
 - transform bounded variables
 - transform heteroskedastic results
 - highlight key results; label everything

5. Fit: dichotomous variables

- (a) Sort estimated probabilities into bins of say 0.1 width: $[0, 0.1)$, $[0.1, 0.2), \dots, [0.9, 1]$.
- (b) From the observations in each bin, compute (a) the mean predictions (probably near 0.05, 0.15, etc.) and (b) the average fraction of 1s.
- (c) Plot (a) by (b) and look for systematic deviation from 45° line.



In sample calibration **graph on right** (from Gary King and Langche Zeng. "Improving Forecasts of State Failure," World Politics, Vol. 53, No. 4 (July, 2001): 623-58)



Out of sample calibration graph on right.

Grouped Uncorrelated Binary Variables

Same model as binary logit, but we only observe sums of iid groups of Bernoulli trials. E.g., the number of times you voted out of the last 5 elections.

$$Y_i \sim \text{Binomial}(y_i | \pi_i) = \binom{N_i}{y_i} \pi_i^{y_i} (1 - \pi_i)^{N_i - y_i}$$

where

$$\pi_i = [1 + e^{-x_i \beta}]^{-1}$$

which implies

$$E(Y_i) \equiv \mu_i = N_i \pi_i = N_i [1 + e^{-x_i \beta}]^{-1}$$

and a likelihood of

$$\begin{aligned} L(\pi | y) &\propto \prod_{i=1}^n \text{Binomial}(y_i | \pi_i) \\ &= \prod_{i=1}^n \binom{N_i}{y_i} \pi_i^{y_i} (1 - \pi_i)^{N_i - y_i} \end{aligned}$$

Grouped Uncorrelated Binary Variables

The Log-likelihood is then:

$$\ln L(\pi|y) = \sum_{i=1}^n \left\{ \ln \binom{N_i}{y_i} + y_i \ln \pi_i + (N_i - y_i) \ln(1 - \pi_i) \right\}$$

and after substituting in the systematic component:

$$\begin{aligned} \ln L(\beta|y) &\doteq \sum_{i=1}^n \left\{ -y_i \ln[1 + e^{-x_i\beta}] + (N_i - y_i) \ln \left(1 - [1 + e^{-x_i\beta}]^{-1} \right) \right\} \\ &= \sum_{i=1}^n \left\{ (N_i - y_i) \ln(1 + e^{x_i\beta}) - y_i \ln(1 + e^{-x_i\beta}) \right\} \end{aligned}$$

Grouped Uncorrelated Binary Variables

Notes:

1. Similar log-likelihood to binary logit
2. All inference is about the same π as in binary logit
3. How to simulate and compute quantities of interest?
 - (a) Run `optim`, and get $\hat{\beta}$ and the variance matrix.
 - (b) Draw many values of $\tilde{\beta}$ from the multivariate normal with mean vector $\tilde{\beta}$ and the variance matrix that come from `optim`.
 - (c) Set X to your choice of values, X_c
 - (d) Calculate simulations of the probability that any of the component binary variables is a one:

$$\tilde{\pi}_c = [1 + e^{-x_c \tilde{\beta}}]^{-1}$$

- (e) If π is of interest, summarize with mean, SD, CI's, or histogram as needed.
- (f) If simulations of y are needed, go one more step and draw $\tilde{y}$ from $\text{Binomial}(y_i | \pi_i)$

Grouped Correlated Binary Variables

In the binomial-logit model, $V(Y) = \pi_i(1 - \pi_i)/N_i$, with no σ^2 -like parameter to take up slack. The beta-binomial (or extended BB) adds this extra parameter. The model:

$$Y_i \sim f_{ebb}(y_i|\pi_i, \gamma)$$

where, recall

$$\begin{aligned} f_{ebb}(y_i|\pi_i, \gamma) &= \Pr(Y_i = y_i|\pi_i, \gamma, N) \\ &= \frac{N!}{y_i!(N - y_i)!} \prod_{j=0}^{y_i-1} (\pi_i + \gamma j) \prod_{j=0}^{N-y_i-1} (1 - \pi_i + \gamma j) / \prod_{j=0}^{N-1} (1 + \gamma j) \end{aligned}$$

and

$$\pi_i = \frac{1}{1 + e^{-x_i\beta}}$$

The probability model of all the data:

$$\begin{aligned}
 \Pr(Y = y|\beta, \gamma; N) &= \prod_{i=1}^n \left(\frac{N!}{y_i!(N - y_i)!} \right) \\
 &\times \prod_{j=0}^{y_i-1} \left\{ [1 + \exp(-x_i\beta)]^{-1} + \gamma_j \right\} \\
 &\times \prod_{j=0}^{N-y_i-1} \left\{ [1 + \exp(x_i\beta)]^{-1} + \gamma_j \right\} / \prod_{j=0}^{N-1} (1 + \gamma_j) \\
 \ln L(\beta, \gamma|y) &= \sum_{i=1}^n \left\{ \ln \left(\frac{N!}{y_i!(N - y_i)!} \right) \right. \\
 &+ \sum_{j=0}^{y_i-1} \ln \left\{ [1 + \exp(-x_i\beta)]^{-1} + \gamma_j \right\} \\
 &+ \left. \sum_{j=0}^{N-y_i-1} \ln \left\{ [1 + \exp(x_i\beta)]^{-1} + \gamma_j \right\} - \sum_{j=0}^{N-1} \ln(1 + \gamma_j) \right\} \\
 &\doteq \sum_{i=1}^n \left\{ \sum_{j=0}^{y_i-1} \ln \left\{ [1 + \exp(-x_i\beta)]^{-1} + \gamma_j \right\} \right. \\
 &+ \left. \sum_{j=0}^{N-y_i-1} \ln \left\{ [1 + \exp(x_i\beta)]^{-1} + \gamma_j \right\} - \sum_{j=0}^{N-1} \ln(1 + \gamma_j) \right\}
 \end{aligned}$$

Notes:

1. The math *looks* complicated.
2. The use of this model is simple.
3. γ soaks up binomial misspecification
4. Assuming binomial when EBB is the right model causes se's to be wrong.
5. How to simulate to compute quantities of interest?
 - (a) Run `optim`, and get $\hat{\beta}$, $\hat{\gamma}$ and the variance matrix.
 - (b) Draw many values of $\tilde{\beta}$ and $\tilde{\gamma}$ from the multivariate normal with mean vector $\hat{\beta}$, $\hat{\gamma}$ and the variance matrix that come from `optim`.
 - (c) Set X to your choice of values, X_c
 - (d) Calculate simulations of the probability that any of the component binary variables is a one:

$$\tilde{\pi}_c = [1 + e^{-x_c \tilde{\beta}}]^{-1}$$

- (e) If π is of interest, summarize with mean, SD, CI's, or histogram as needed.
- (f) If simulations of y are needed, go one more step and draw $\tilde{y}$ from $f_{ebb}(y_i | \pi_i)$

Event Count Models: Poisson

Uses:

1. The number of cooperative and conflictual international incidents,
2. The number of triplets born in Norway in each half-decade
3. The annual number of presidential appointments to the Supreme Court
4. The number of Coups d'Etat in black African states
5. The number of medical consultations for each survey respondent
6. For each of these examples, the upper limit on the number of observed events is theoretically infinite.
7. Some event count datasets go over time (count per year), some across areas (count per state), and some both.

Recall Poisson distribution's first principles:

1. Begin with an observation period and count point:
2. Assumptions are about: events occurring between start and count observation. The process of event generation is assumed not observed.
3. 0 events occur at the start of the period
4. Observe only: number of events at end of the period
5. No 2 events can occur at the same time
6. $\Pr(\text{event at time } t \mid \text{all events up to time } t - 1)$ is constant for all t .

The Poisson regression model:

$$Y_i \sim \text{Poisson}(y_i | \lambda_i)$$

$$\lambda_i = \exp(x_i \beta)$$

and, as usual, Y_i and Y_j are independent $\forall i \neq j$, conditional on X .

The probability density of all the data:

$$P(y|\lambda) = \prod_{i=1}^n \frac{e^{-\lambda_i} \lambda_i^{y_i}}{y_i!}$$

The log-likelihood:

$$\begin{aligned} \ln L(\beta|y) &= \sum_{i=1}^n \{y_i \ln(\lambda_i) - \lambda_i - \ln(y_i!)\} \\ &= \sum_{i=1}^n \{(x_i \beta) y_i - \exp(x_i \beta) - \ln y_i!\} \\ &\doteq \sum_{i=1}^n \{(x_i \beta) y_i - \exp(x_i \beta)\} \end{aligned}$$

Poisson Model Interpretation

1. Derivative method:

$$\frac{\partial \lambda_i}{\partial X_i^1} = \exp(x_i \beta) \beta = \lambda_i \beta$$

so we could use $\bar{y}\beta$ for an approximate linearized effect.

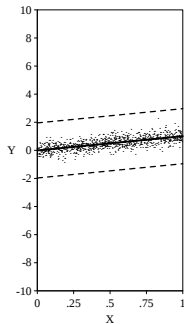
2. To simulate:

- (a) Set X_c
- (b) Draw $\tilde{\beta}$ from $N(b|\hat{\beta}, \hat{V}(\hat{\beta}))$
- (c) Compute $\tilde{\lambda}_c = \exp(X_c \tilde{\beta})$
- (d) Draw Y_c from $\text{Poisson}(y|\tilde{\lambda})$

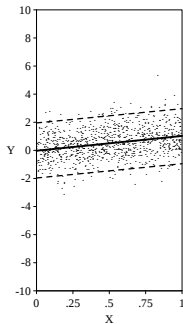
3. Under the Poisson: $V(Y_i|X_i) = E(Y_i|X_i)$, which is heteroskedastic and fixed.

- (a) Level of dispersion is conditional on X , so it changes with the specification
- (b) $V(Y_i|X_i) > E(Y_i|X_i)$ is overdispersion: standard errors will be too small (very common)
- (c) $V(Y_i|X_i) < E(Y_i|X_i)$ is underdispersion: standard errors will be too big

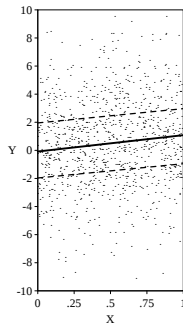
What happens without an extra parameter? Stylized Normal.



(a)



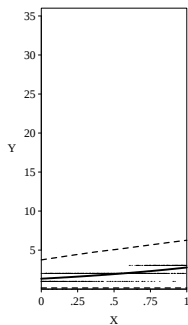
(b)



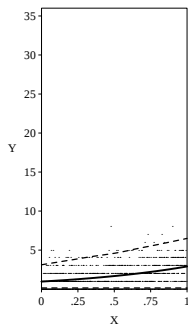
(c)

$E(Y|X)$ and 95% CI.

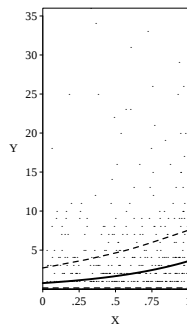
What happens without an extra parameter? Poisson



(a)



(b)



(c)

$E(Y|X)$ and 95% CI.

Negative Binomial Event Count Model

For overdispersed data (conditional on X), the model:

$$Y_i \sim \text{NegBin}(y_i | \phi, \sigma^2)$$
$$E(Y_i) \equiv \phi = \exp(x_i \beta)$$

Interpretation:

1. $V(Y|X) = \phi\sigma^2$, and $\sigma^2 > 1$
2. Recall:

$$\lim_{\sigma^2 \rightarrow 1} \text{Negbin}(y_i | \phi_i, \sigma^2) = \text{Poisson}(y_i | \phi_i)$$

3. Test of Poisson vs NegBin: look at σ^2 (likelihood ratio doesn't work since Poisson doesn't exactly nest within Negbin)

The Negative Binomial Likelihood

The probability density of all the data:

$$P(y|\phi, \sigma^2) = \prod_{i=1}^n \frac{\Gamma\left(\frac{\phi}{\sigma^2-1} + y_i\right)}{y_i! \Gamma\left(\frac{\phi}{\sigma^2-1}\right)} \left(\frac{\sigma^2-1}{\sigma^2}\right)^{y_i} (\sigma^2)^{\frac{-\phi}{\sigma^2-1}}$$

The log-likelihood:

$$\ln L(\phi, \sigma^2|y) = \sum_{i=1}^n \left\{ \ln \Gamma\left(\frac{\phi}{\sigma^2-1} + y_i\right) - \ln y_i! \right. \\ \left. - \ln \Gamma\left(\frac{\phi}{\sigma^2-1}\right) + y_i \ln\left(\frac{\sigma^2-1}{\sigma^2}\right) - \left(\frac{\phi}{\sigma^2-1}\right) \ln(\sigma^2) \right\}$$

Computational Issues

1. $\ln \Gamma(a)$ with large a is hard to compute in 2 steps (since $\Gamma(a) \approx a!$ is immense) but easy in one. In R, see `lgamma`.
2. β is unbounded as is, and so no need to reparameterize.
3. $\sigma^2 > 1$, and so we estimate γ , where $\sigma^2 = e^\gamma + 1$.

A Generalized Event Count (GEC) Model

An event count model with under-, Poisson, and over-dispersion

Stochastic component:

$$\begin{aligned} Y_i &\sim \text{GEC}(y_i|\lambda_i, \sigma^2) \equiv P(Y = y_i|\lambda_i, \sigma^2) \\ &= \text{GEC}\left(y_i|e^{X_i\beta}, \sigma^2\right) \\ &= \frac{1}{y_i!} \left(\frac{e^{X_i\beta}}{\sigma^2}\right)^{\left(y_i, 1-\frac{1}{\sigma^2}\right)} \left[\sum_{j=0}^{y_i^{\max}} \frac{1}{j!} \left(\frac{e^{X_i\beta}}{\sigma^2}\right)^{\left(j, 1-\frac{1}{\sigma^2}\right)}\right]^{-1}, \end{aligned}$$

where $y_i^{\max} = \infty$ for $\sigma^2 \geq 1$, $n_i = \lambda_i/(1 - \sigma^2)$, $y_i^{\max} = [n_i + 1]$ for $0 < \sigma^2 < 1$, and $[x] = x - 1$ for integer x and floor(x) for non-integer x .

$$x^{(m,\delta)} = \begin{cases} \prod_{i=0}^{m-1} (x + \delta i) = x(x + \delta)(x + 2\delta) \cdots [x + \delta(m - 1)] & m \geq 1 \\ 1 & m = 0 \end{cases}$$

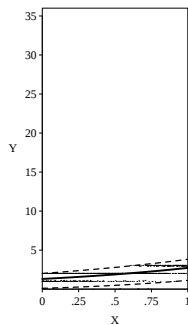
Systematic component:

$$E(Y_i|X_i) \equiv \lambda_i = \exp(X_i\beta)$$

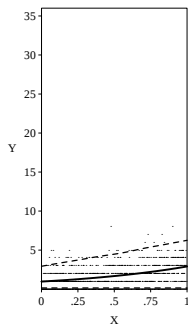
GEC Interpretation:

1. Special cases of the GEC
 - (a) *Negative Binomial*, $\sigma^2 > 1$, the over-dispersed case.
 - (b) *Poisson*, $\sigma^2 = 1$
 - (c) *Continuous Parameter Binomial*, $0 < \sigma^2 < 1$, the underdispersed case.
This special case itself reduces to an even more special case, the *Binomial*, when $\lambda_i/(1 - \sigma^2)$ is an integer.
2. Can simulate in three parts, indexed by σ^2 .

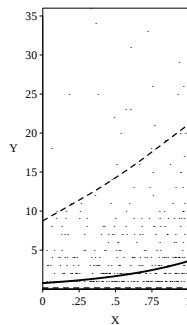
What happens with the extra parameter? GEC



(a)



(b)



(c)

$E(Y|X)$ and 95% CI.

King, Gary and Curtis S. Signorino. "The Generalization in the Generalized Event Count Model" in *Political Analysis*, 6 (1996): 225-252.

King, Gary. "Variance Specification in Event Count Models: From Restrictive Assumptions to a Generalized Estimator," *American Journal of Political Science*, 33, 3 (August, 1989): 762-784.

Duration Models and Censoring: The Exponential Model

King, Gary; James Alt; Nancy Burns; and Michael Laver. "A Unified Model of Cabinet Dissolution in Parliamentary Democracies," *American Journal of Political Science*, Vol. 34, No. 3 (August, 1990): Pp. 846-871; Errata Vol. 34, No. 4 (November, 1990): P. 1168. (replication dataset: ICPSR s1115).

Duration Models and Censoring: The Exponential Model

A density with the same first principles as the Poisson, except that we observe Y , the duration between events. Then

$$Y_i \sim \text{expon}(y_i | \lambda_i) = \lambda_i e^{-\lambda_i y_i}$$

So the exponential duration model:

$$Y_i \sim \text{expon}(y_i | \lambda_i)$$
$$E(Y_i) \equiv \frac{1}{\lambda_i} = \frac{1}{e^{-x_i \beta}} = e^{x_i \beta}$$

Log-likelihood:

$$\begin{aligned} \ln L(\beta | y) &= \sum_{i=1}^n \{\ln \lambda_i - \lambda_i y_i\} \\ &= \sum_{i=1}^n \left\{ -x_i \beta - e^{-x_i \beta} y_i \right\} \end{aligned}$$

What do we do with observations that have not yet ended?

1. For example
 - (a) Parliamentary coalition duration; some still in office
 - (b) Duration of unemployment spells; some people still unemployed
 - (c) Duration in graduate school. (What will we do with you?)
2. Could drop them. Selection bias.
3. Could set duration to observed level: would underestimate in most cases
4. Could guess. Even a good guess would produce standard errors that are too small.
5. We will model these observations explicitly as censored.

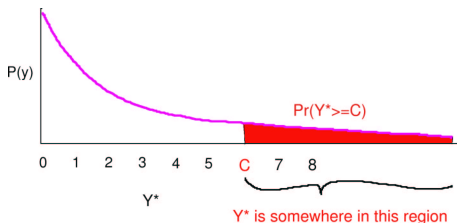
Censoring

$$Y_i^* \sim \text{expon}(y_i^* | \lambda_i)$$
$$E(Y_i^*) \equiv \frac{1}{\lambda_i} = \frac{1}{e^{-x_i\beta}} = e^{x_i\beta}$$

An observation mechanism (with censoring at C):

$$y_i = \begin{cases} y_i^* & \text{if } y_i^* < C \\ y_i^C & \text{if } y_i^* \geq C \end{cases}$$

The likelihood function for censored observations. All we know is:



$$\begin{aligned}\Pr(Y_i = y_i^c) &= \Pr(Y_i^* \geq y_i^c) \\ &= \int_{y_i^c}^{\infty} \text{expon}(y_i | \lambda_i) dy_i \\ &= \int_{y_i^c}^{\infty} \lambda_i e^{-\lambda_i y_i} dy_i \\ &= e^{-\lambda_i y_i^c}\end{aligned}$$

Thus, the full likelihood:

$$L(\beta|y) = \left[\prod_{y_i^* < y_i^c} \text{expon}(y_i | \lambda_i) \right] \left[\prod_{y_i^* \geq y_i^c} \Pr(Y_i^* \geq y_i^c) \right]$$