Advanced Quantitative Research Methodology, Lecture Notes: Multiple Equation Models[1]

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April 20, 2013

Identification

Reading: Gary King. *Unifying Political Methodology: The Likelihood Theory of Statistical Inference*. Ann Arbor: University of Michigan Press, 1998: Chapter 8.

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Models that often don't make sense, even though it is hard to tell.

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3. A likelihood with a plateau can be informative, but a unique MLE doesn't exist

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So $\{\beta_2 = 2, \beta_3 = 5\}$ gives the same likelihood as $\{\beta_2 = 5, \beta_3 = 2\}$.

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(BTW, you now know how to form the likelihood for multiple equation models!)

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Also assume parametric independence, and you can estimate the equations separately.

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$$\begin{aligned} L(\beta, \Sigma) &= \prod_{i=1}^{n} N(y_i|\mu_i, \Sigma) \\ &= \prod_{i=1}^{n} (2\pi)^{-1} |\Sigma|^{-1/2} \exp\left[-\frac{1}{2}(y_i - \mu_i)'\Sigma^{-1}(y_i - \mu_i)\right] \end{aligned}$$

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7. $\implies$ identification of extra parameters in multiple equation models with identical X's comes solely from model assumptions

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$$\begin{aligned} \text{Vote:} \quad & \mu_{1i} = x_{1i}\beta_1 + x_{2i}\beta_2 + \mu_{2i}\beta_3 \\ \text{PID:} \quad & \mu_{2i} = x_{1i}\gamma_1 + x_{3i}\gamma_2 + \mu_{1i}\gamma_3 \end{aligned}$$

Where,
x_1 demographics

Reciprocal Causation

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x_3 parents PID (affecting PID but not vote)

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f(y|\mu,\Sigma) &= \prod_{i=1}^{n} N(y_i|\mu_i,\Sigma) \\
&= \prod_{i-1}^{n} (2\pi)^{N/2}|\Sigma|^{-1/2} \exp\left\{-\frac{1}{2}(y_i-\mu_i)'\Sigma^{-1}(y_i-\mu_i)\right\}
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Where,

$$
\begin{aligned}
y_i &= (y_{1i}, y_{2i})', \\
\mu_i &= (\mu_{1i}, \mu_{2i})' \\
\Sigma &= \begin{pmatrix} \sigma_1 & \sigma_{12} \\ \sigma_{12} & \sigma_2 \end{pmatrix}
\end{aligned}
$$

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1. Standard models are reparameterized
2. Time series models are recursively reparameterized
3. This model requires *multiple equation reparameterization*.

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- $\{\beta_1 = 1, \gamma_1 = 30, \beta_3 = 555, \gamma_3 = -30\} \implies \mu_{1i} = -x_i$.
- $\{\beta_1 = 1, \gamma_1 = 1, \beta_3 = -999, \gamma_3 = -1\} \implies \mu_{1i} = -x_i$.
- As long as $\beta_1 = 1$ and $\gamma_1 = -\gamma_3$, for any value of β_3, $\mu_{1i} = -x_i$.

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- All results are highly sensitive to X_2 and X_3

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$$\mu_{ij} = x_{ij}\beta_j$$

with observation mechanism:

$$Y_{ij} = \begin{cases} 1 & \text{if } U_{ij}^* > U_{ij'}^* \; \forall \; j \neq j' \\ 0 & \text{otherwise} \end{cases}$$

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$$Y_{ij} = \begin{cases} 1 & \text{if } Y^*_{ij} > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \pi_{ij} &= \Pr(y_{ij} = 1) \\ &= \Pr(Y^*_{i1} \leq 0, \ldots, {\color{red} Y^*_{ij} > 0}, \ldots, Y^*_{iJ} \leq 0) \end{aligned}$$

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$$\begin{aligned} \pi_{ij} &= \Pr(y_{ij} = 1) \\ &= \Pr(Y_{i1}^* \leq 0, \ldots, Y_{ij}^* > 0, \ldots, Y_{iJ}^* \leq 0) \\ &= \int_{-\infty}^{0} \cdots \int_{0}^{\infty} \cdots \int_{-\infty}^{0} N(y|\mu_i, \Sigma) dy_{i1} \cdots dy_{ij} \cdots dy_{iJ} \end{aligned}$$

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5. The entire model is only weakly identified
6. Model is a straightforward generalization of SURM, or of univariate probit

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4. Applying binary logit: telling the computer you have $2n$ observations (husbands and wives) when you really only have n (families) — known as the double barreled data extender!

5. We instead want the *conditional* probability (i.e., the prob it is the husband given that it is either the husband or wife) and so use the usual rule: $\Pr(A|B) = \Pr(AB)/\Pr(B)$. Thus,

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&= \frac{\left(\frac{e^{X_{i1}\beta}}{1 + e^{X_{i1}\beta}}\right)\left(\frac{1}{1 + e^{X_{i2}\beta}}\right)}{\left(\frac{e^{X_{i1}\beta}}{1 + e^{X_{i1}\beta}}\right)\left(\frac{1}{1 + e^{X_{i2}\beta}}\right) + \left(\frac{1}{1 + e^{X_{i1}\beta}}\right)\left(\frac{e^{X_{i2}\beta}}{1 + e^{X_{i2}\beta}}\right)}
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5. Say again? This makes sense since we're asking (e.g.,) whether the husband or wife does the checkbook in each marriage. We don't have any variation (given this question) whether the husband from marriage 1 has a higher probability than the husband from marriage 2.

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   (a) Reading: King, Gary. "Proper Nouns and Methodological Propriety: Pooling Dyads in International Relations Data," [concluding comment in a symposium on the analysis of dyadic international conflict data, with papers by Donald Green, Soo Yeon Kim, and David Yoon; John Oneal and Bruce Russett; and Nathaniel Beck and Jonathan Katz], *International Organization*, Vol. 55, No. 2 (Fall, 2001): Pp. 497–507.

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(k) The alternative is used, but its not a very happy solution!

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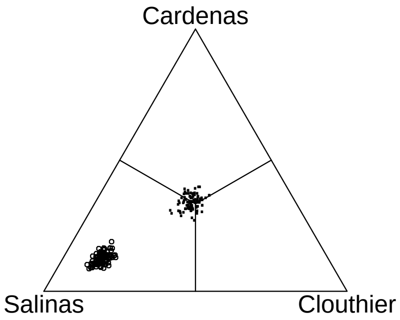
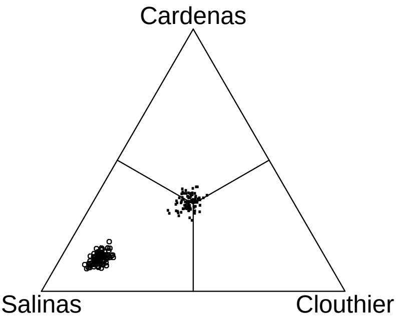
$$\pi_i = \frac{e^{X_i\beta_j}}{\sum_{k=1}^{3} e^{X_i\beta_k}} \qquad \text{where } j = 1, 2, 3 \text{ candidates.}$$


Cardenas
Salinas
Clouthier



5. Each point in the figure is an election outcome drawn randomly from a world in which all voters believe Salinas' PRI party is strengthing (for the "o"'s in the bottom left) or weakening (for the "."'s in the middle), with other variables held constant at their means. (100 of each).

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