

Advanced Quantitative Research Methodology, Lecture Notes: Theories of Inference¹

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The Problem of Inference

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3. A more reasonable, limited goal. Let $M = \{M^*, \theta\}$, where M^* is assumed & θ is to be estimated:

$$P(\theta|y, M^*) \equiv P(\theta|y)$$

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8. The two differ on the rest

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6. $L(\theta|y)$ is a function: for y fixed at the observed values, it gives the “likelihood” of any value of θ .

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9. Comparing values of $L(\theta|y)$ across data sets is meaningless. (just as you can't compare R^2 values across equations with different dependent variables.)
10. The **likelihood principle**: the data only affect inferences through the likelihood function

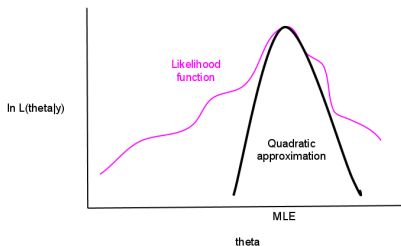
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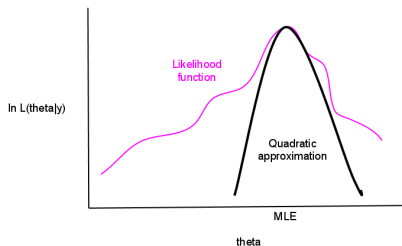
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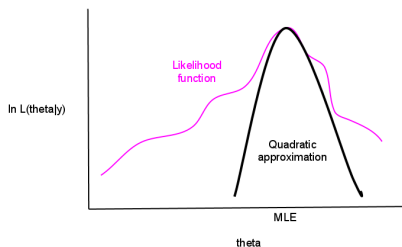
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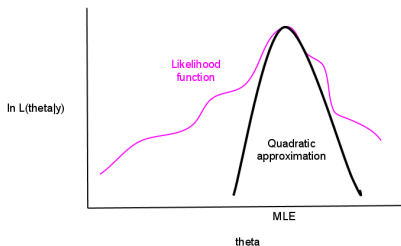
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- Uncertainty of point estimate: **curvature at the maximum**

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- $P(\theta)$, the prior density — the way Bayes differs from likelihood

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4. A **philosophical assumption** that nonsample information should matter (as it always does) *and* be formalized and included in all inferences.

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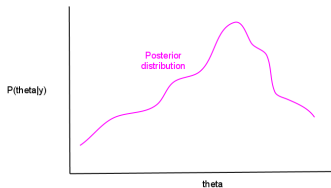
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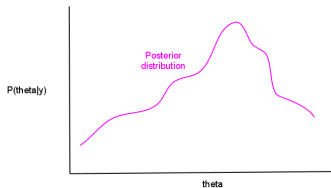
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4. If variables A and B are both unknown, then the distribution of A alone is $P(A) = \int P(A, B)dB = \int P(A|B)P(B)dB$.

The posterior density, $P(\theta|y)$

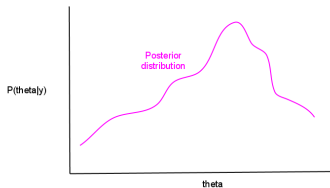


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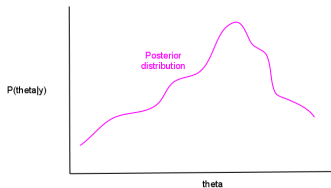
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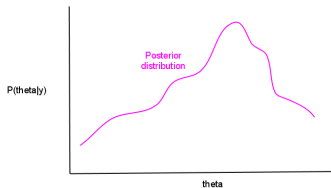
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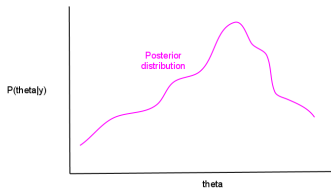
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- If $P(\theta) = 1$, i.e., is uniform in the relevant region, then $L(\theta|y) = P(\theta|y)$.

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- Few fights now between Bayesians and likelihoodists

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- Assume n is large enough for the CLT to kick in

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A 3rd Theory: Neyman-Pearson Hypothesis Testing

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- or

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Neyman-Pearson Hypothesis Testing

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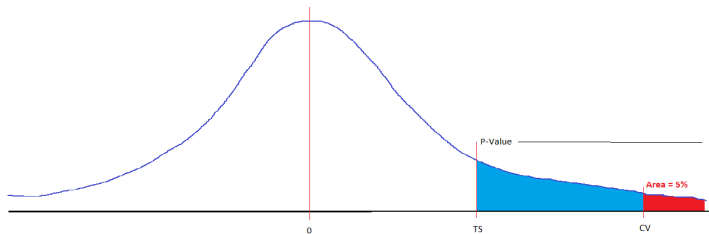
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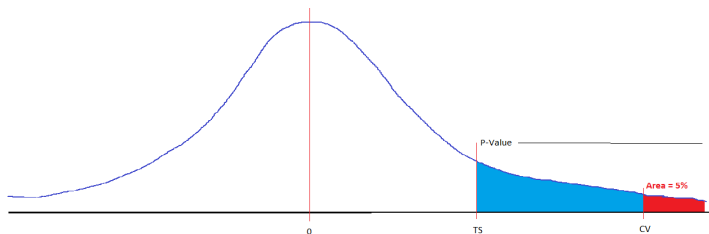
And then first collect your data. You may not revise your hypothesis or your theory.

Neyman-Pearson Hypothesis Testing

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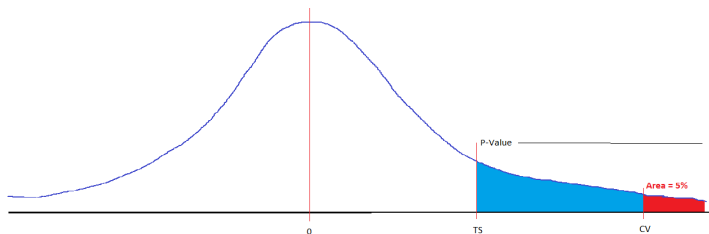


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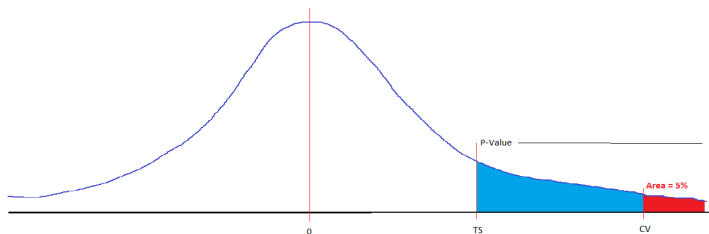
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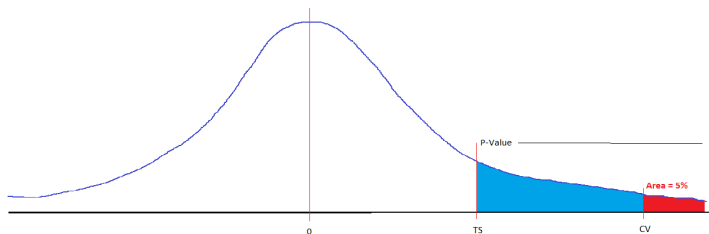
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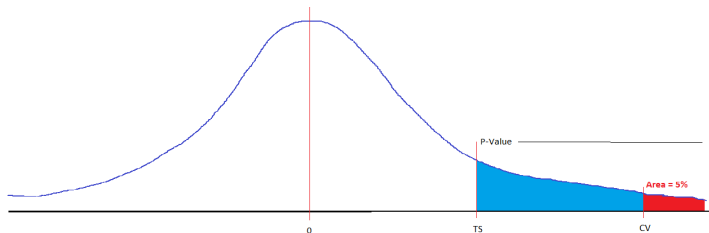
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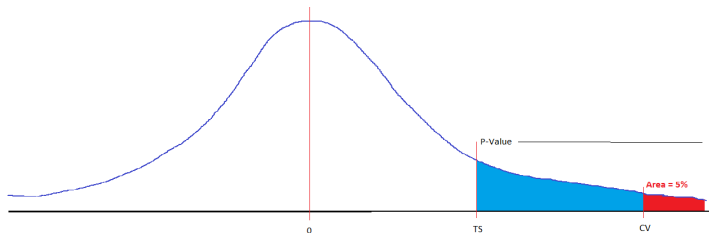
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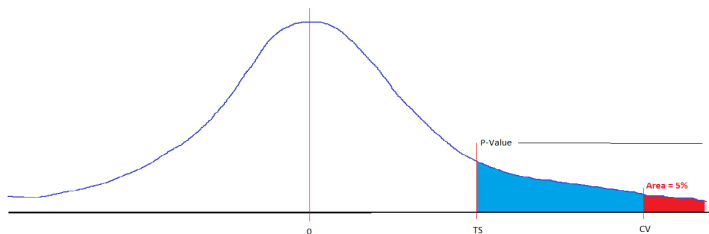
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4. Methods for applied researchers: either useful or irrelevant

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- The key: No assumptions can be trusted; all theories of inference condition on assumptions and so data analysts always struggle trying to understand and get around them

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- What can you do with this probability density?

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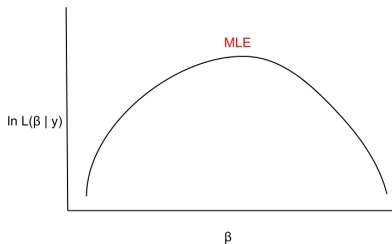
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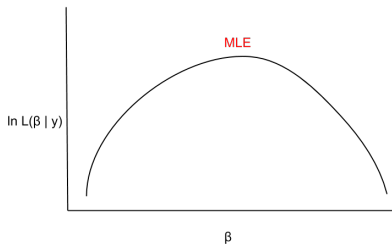
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Log-likelihood interpretation

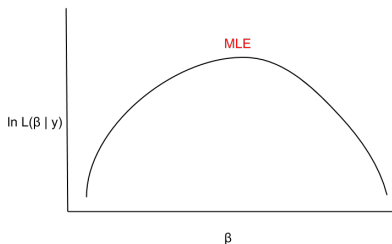


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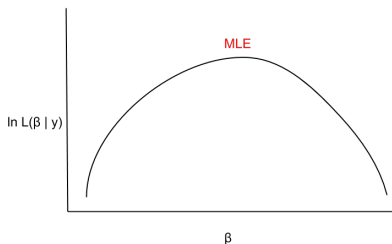
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Log-likelihood interpretation



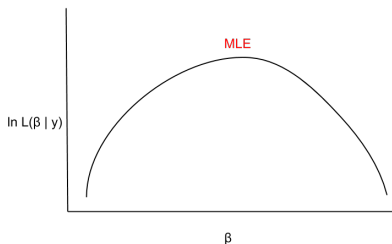
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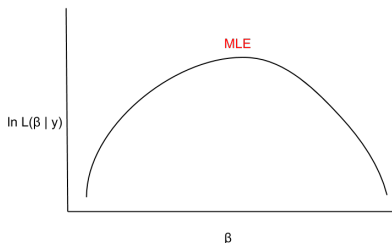
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5. No reason to summarize this curve with only the MLE

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- The curvature at the maximum (standard errors, about which more shortly)

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- We'll show you how

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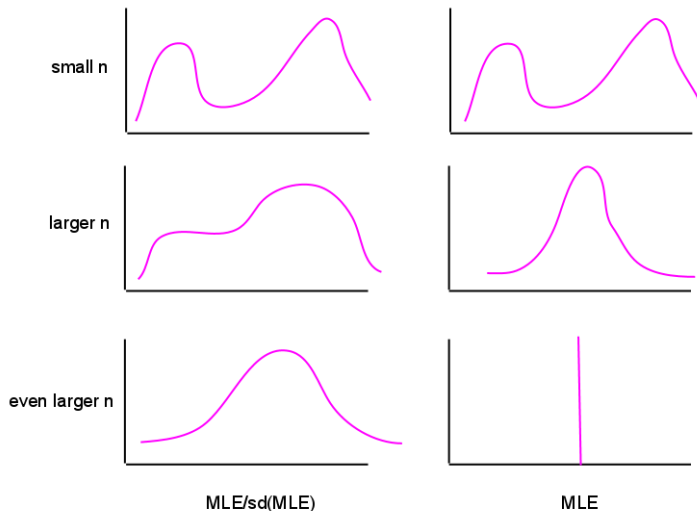
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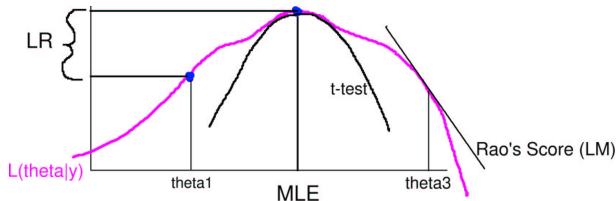
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- ③ **Asymptotic efficiency**. The MLE contains as much information as can be packed into a point estimator.

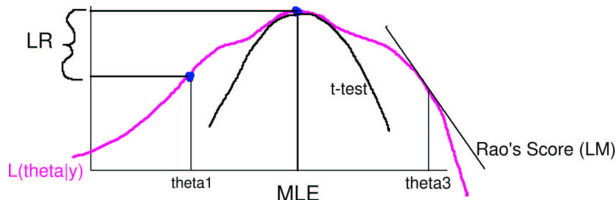
Sampling distributions of the MLE: CLT vs LLN



Uncertainty: Likelihood Ratios for nested models

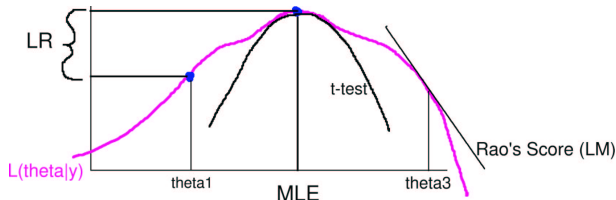


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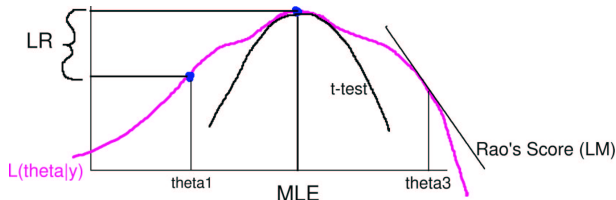
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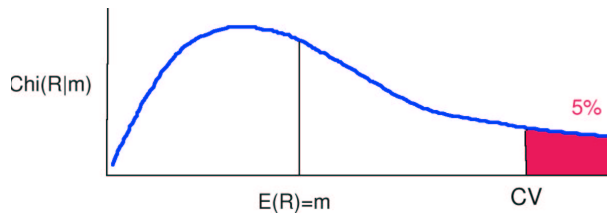
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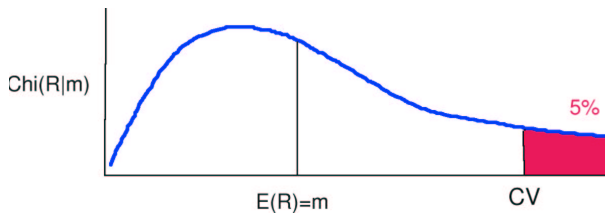
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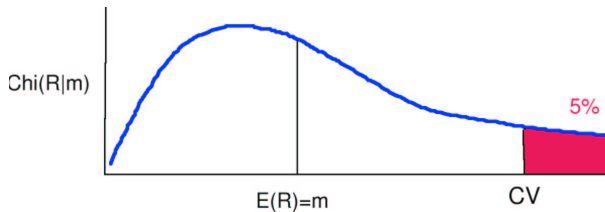


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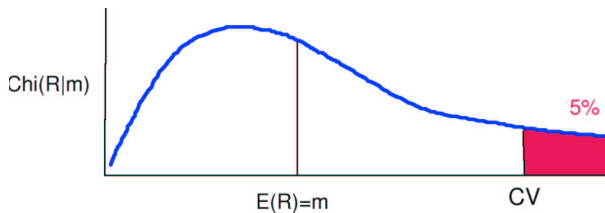
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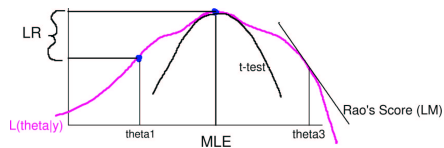
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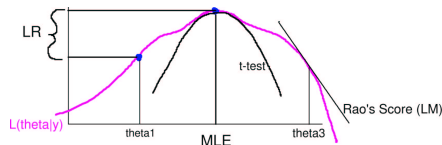
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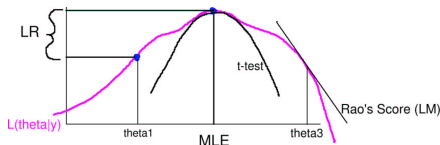


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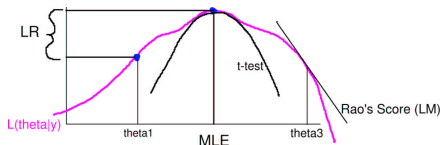
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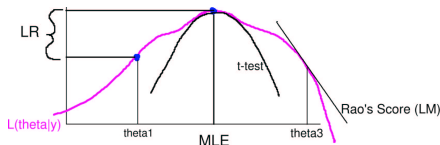
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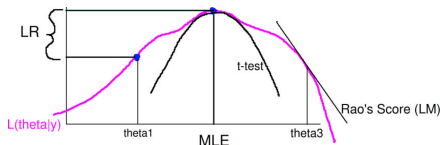
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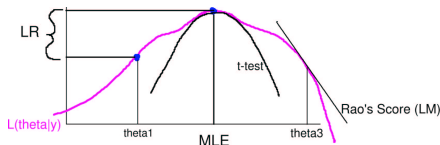
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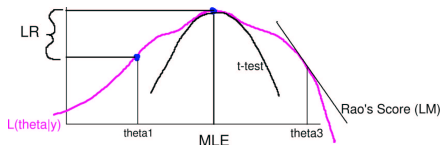
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- Works in general for a k -dimensional θ vector
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Standard Errors

7. When the log-likelihood is not normal, we'll use the best quadratic approximation to it. Under the normal,

$$\frac{\partial^2 \ln L(\beta|y)}{\partial \beta \partial \beta'} = \frac{-n}{\sigma^2}$$

More generally, this second derivative will give us a way to compute the coefficient on the squared term.

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9. This is an **estimate** of a **quadratic approximation** to the log-likelihood.

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Forecasting Presidential Elections.

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E_i	The number of electoral college votes for each state in 2012

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- Compute and save $\hat{V}(\hat{\theta})$, which is $k + 2 \times k + 2$

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- Mathematical Form:

$$\ln L(\beta, \sigma^2 | y) = \sum_{i=1}^n \sum_{t=1}^T -\frac{1}{2} \left[\ln \sigma^2 + \frac{(y_{it} - X_{it}\beta)^2}{\sigma^2} \right]$$

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- An R function:

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ll.normal <- function(par, X, Y) {  
  X <- as.matrix(cbind(1, X))  
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- Calling it:

```
ll.normal(c(2,1,2,1,33,4,3.2),x,y)  
ll.normal(c(2,1,2,1,33,4,3.7),x,y)  
ll.normal(c(2,1,2,1,33,4,3.5),x,y)
```

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- Repeat Steps 1–3 $M = 1,000$ times, and plot a histogram of the results.

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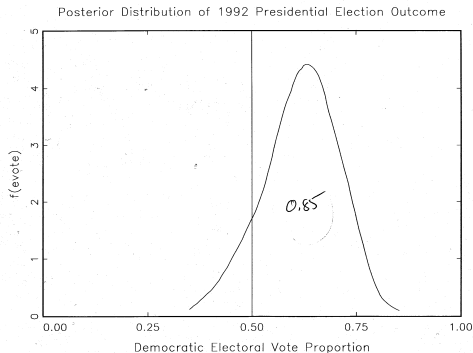
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4. Either:
 - Draw ϵ_{it} from $N(0, \tilde{\sigma}^2)$, label it $\tilde{\epsilon}_{it}$ and compute: $\tilde{y}_{i,2012} = \tilde{X}_{i,2012}\tilde{\beta} + \tilde{\epsilon}_{it}$

How to do it with a LS Regression Program

1. Run LS regression of y_{it} on X_{it} and get $\hat{\beta}$ and $V(\hat{\beta})$
2. Draw β randomly from its posterior distribution (i.e., its sampling distribution), $N(\beta|\hat{\beta}, V(\hat{\beta}))$. Label the random draw $\tilde{\beta}$.
3. Draw σ^2 from its posterior (or sampling) distribution, $1/\chi^2(\hat{\sigma}^2, N - k)$, labeling it $\tilde{\sigma}^2$
4. Either:
 - Draw ϵ_{it} from $N(0, \tilde{\sigma}^2)$, label it $\tilde{\epsilon}_{it}$ and compute: $\tilde{y}_{i,2012} = \tilde{X}_{i,2012}\tilde{\beta} + \tilde{\epsilon}_{it}$
 - Or, in our preferred notation, draw $\tilde{y}_{i,2012}$ from $N(X_{i,2012}\tilde{\beta}, \tilde{\sigma}^2)$

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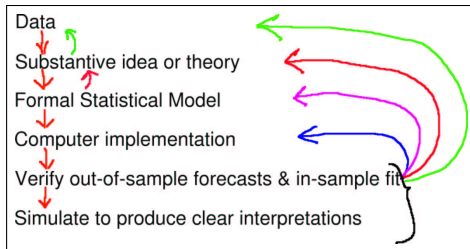
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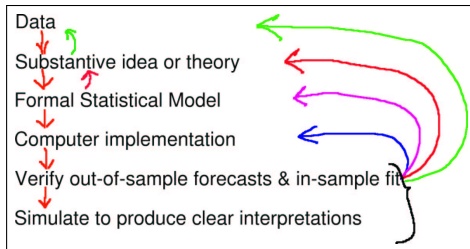
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- For what applications would this model be informative?

An Outline of the Research Process

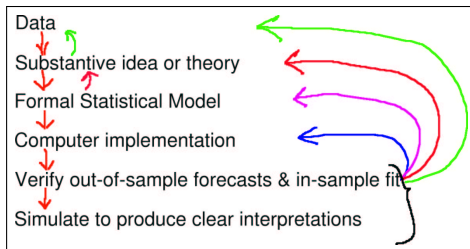


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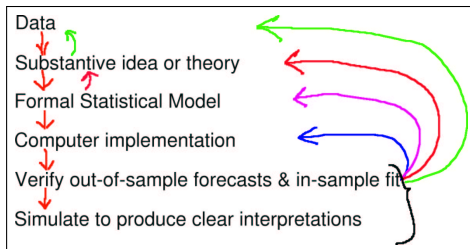
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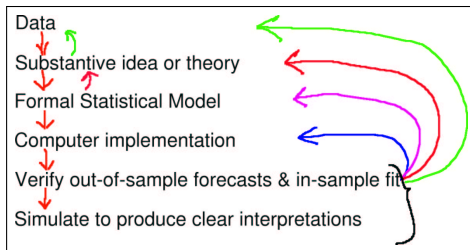
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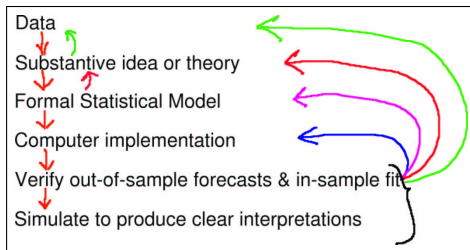
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5. Don't miss any parts.