

Advanced Quantitative Research Methodology, Lecture Notes: Theories of Inference¹

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February 23, 2014

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The Problem of Inference

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3. A more reasonable, limited goal. Let $M = \{M^*, \theta\}$, where M^* is assumed & θ is to be estimated:

$$P(\theta|y, M^*) \equiv P(\theta|y)$$

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8. The two differ on the rest

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6. $L(\theta|y)$ is a function: for y fixed at the observed values, it gives the “likelihood” of any value of θ .

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9. Comparing values of $L(\theta|y)$ across data sets is meaningless. (just as you can't compare R^2 values across equations with different dependent variables.)
10. The **likelihood principle**: the data only affect inferences through the likelihood function

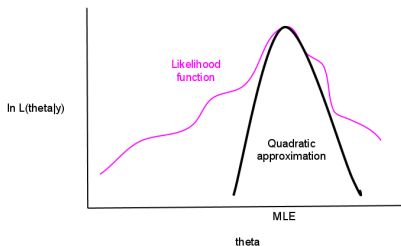
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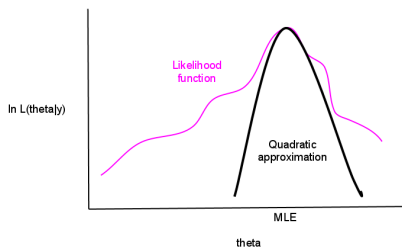
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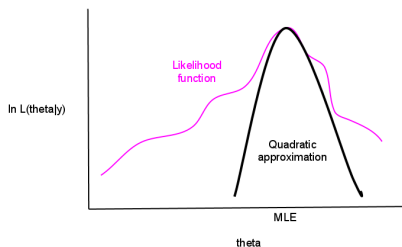
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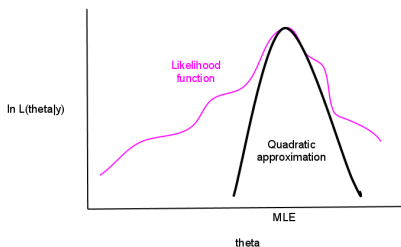
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- Uncertainty of point estimate: **curvature at the maximum**

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- $P(\theta)$, the prior density — the way Bayes differs from likelihood

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4. A **philosophical assumption** that nonsample information should matter (as it always does) *and* be formalized and included in all inferences.

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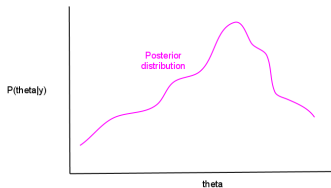
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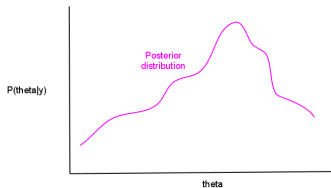
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4. If variables A and B are both unknown, then the distribution of A alone is $P(A) = \int P(A, B)dB = \int P(A|B)P(B)dB$.

The posterior density, $P(\theta|y)$

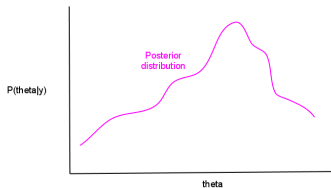


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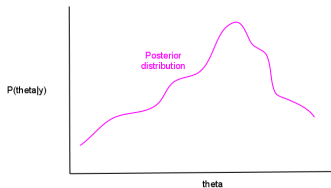
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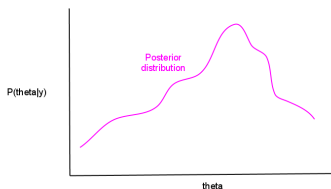
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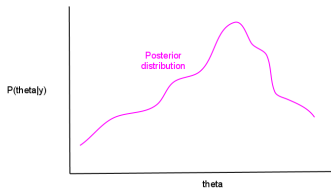
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- If $P(\theta) = 1$, i.e., is uniform in the relevant region, then $L(\theta|y) = P(\theta|y)$.

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- Few fights now between Bayesians and likelihoodists

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- Assume n is large enough for the CLT to kick in

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A 3rd Theory: Neyman-Pearson Hypothesis Testing

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Neyman-Pearson Hypothesis Testing

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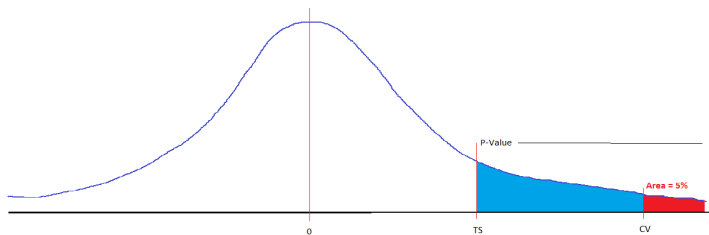
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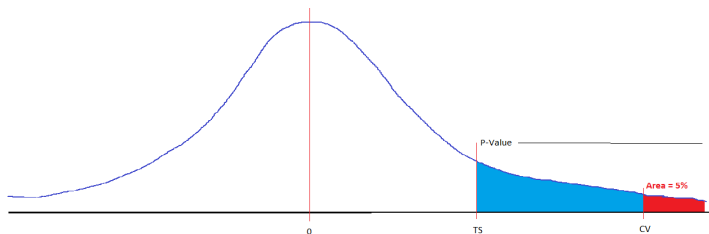
And then first collect your data. You may not revise your hypothesis or your theory.

Neyman-Pearson Hypothesis Testing

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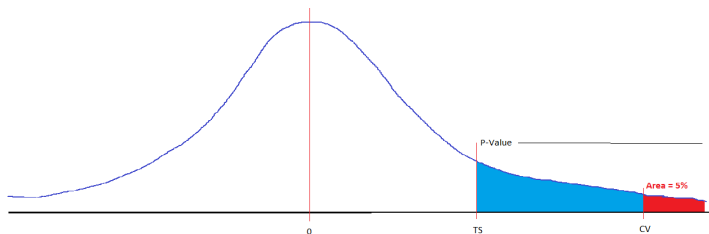


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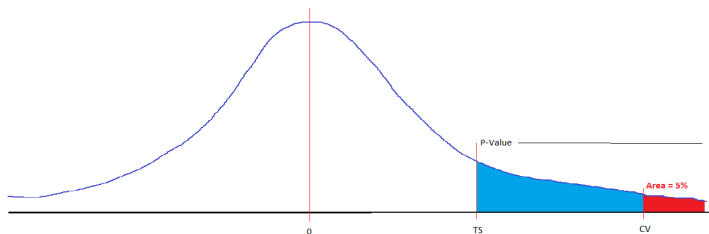
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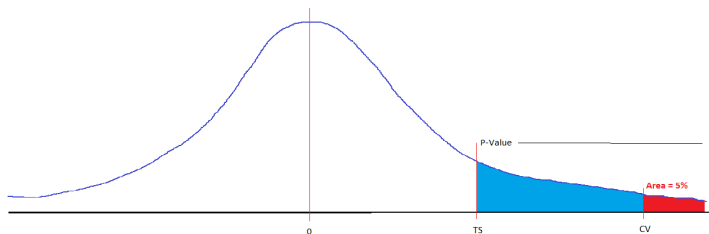
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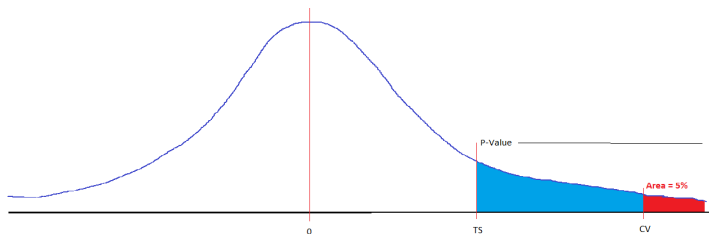
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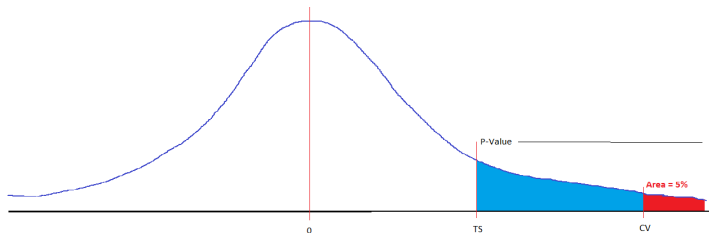
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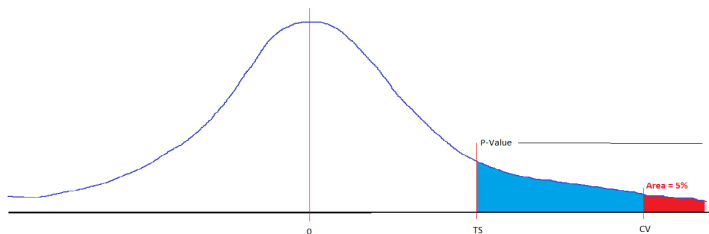
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4. Methods for applied researchers: either useful or irrelevant

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- The key: No assumptions can be trusted; all theories of inference condition on assumptions and so data analysts always struggle trying to understand and get around them

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- What can you do with this probability density?

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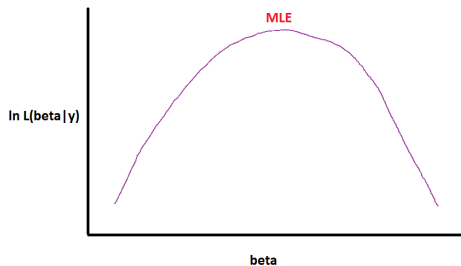
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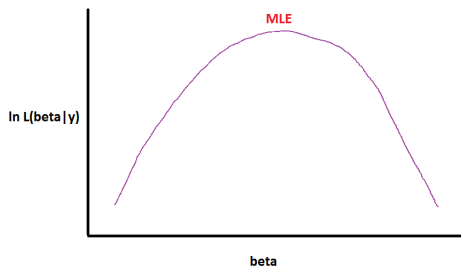
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Log-likelihood interpretation

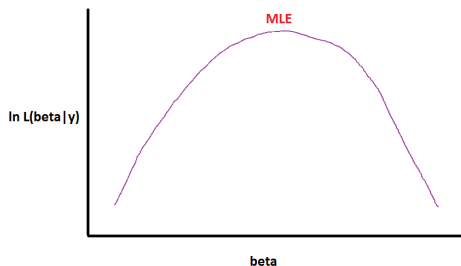


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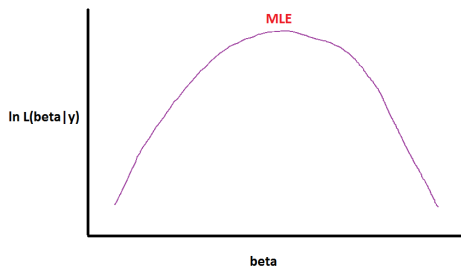
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Log-likelihood interpretation



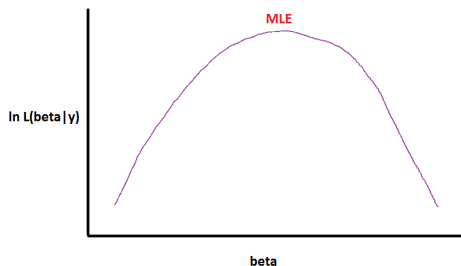
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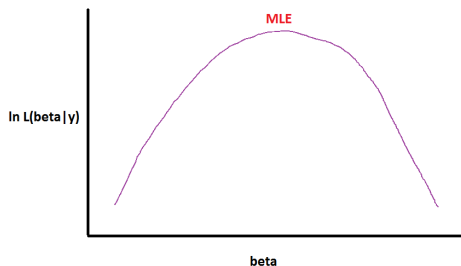
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5. No reason to summarize this curve with only the MLE

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- Graphs

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- The curvature at the maximum (standard errors, about which more shortly)

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- We'll show you how

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- In fact, it's a great idea!

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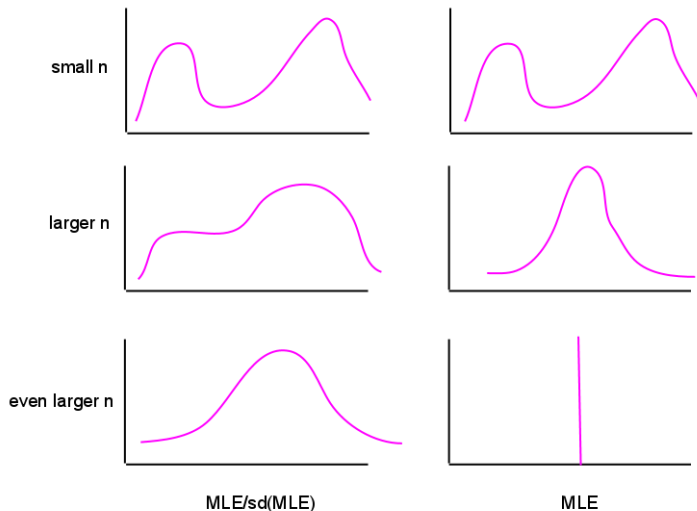
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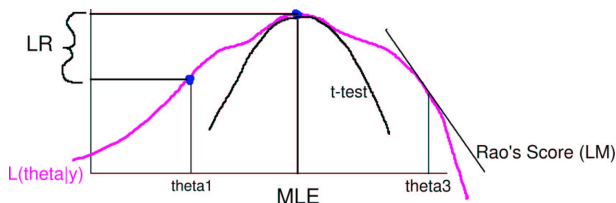
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- ③ **Asymptotic efficiency**. The MLE contains as much information as can be packed into a point estimator.

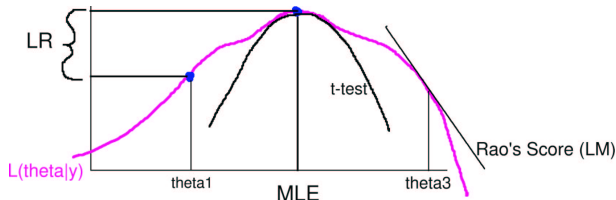
Sampling distributions of the MLE: CLT vs LLN



Uncertainty: Likelihood Ratios for nested models

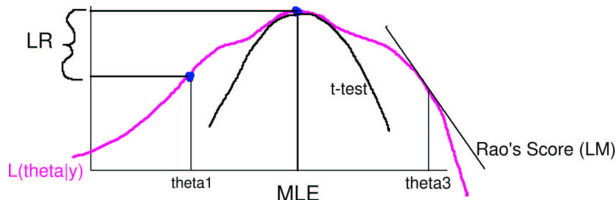


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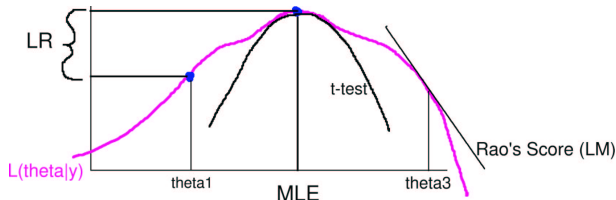
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Interpreted as a *risk ratio*.

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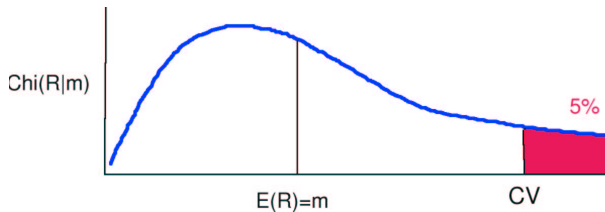
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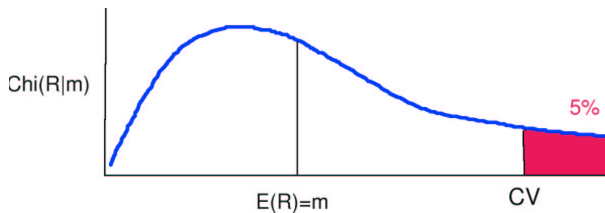
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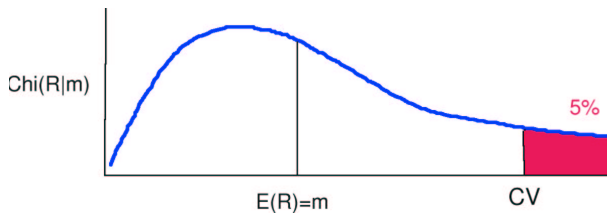


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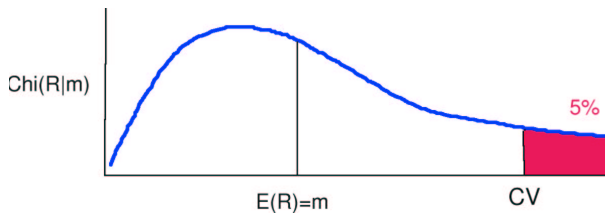
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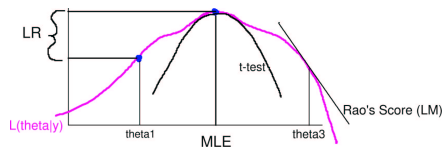
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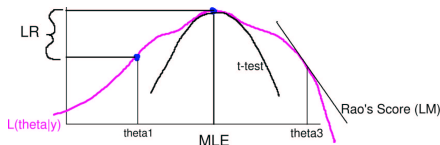
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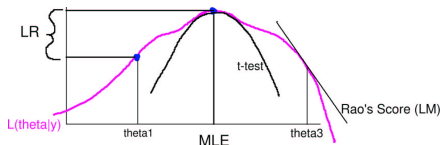


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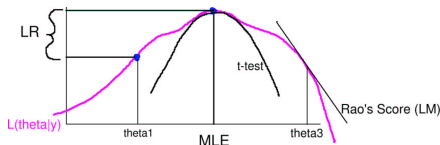
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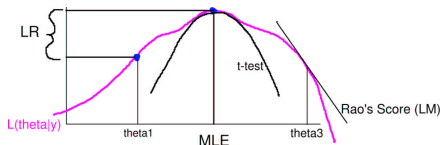
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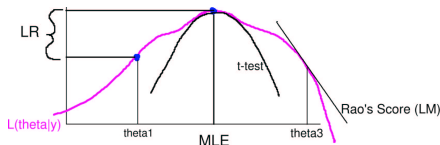
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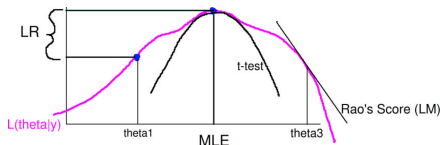
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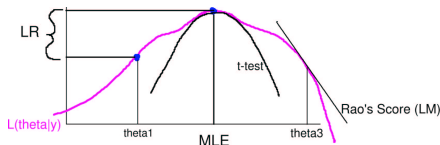
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E_i	The number of electoral college votes for each state in 2012

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$$\begin{aligned} L(\mu_{it}, \sigma | y_{it}) &\propto N(y_{it} | \mu_{it}, \sigma^2) \\ &= (2\pi\sigma^2)^{-1/2} e^{\frac{-(y_{it} - \mu_{it})^2}{2\sigma^2}} \end{aligned}$$

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- Compute and save $\hat{V}(\hat{\theta})$, which is $k + 2 \times k + 2$

R Code for the Log-Likelihood

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- Mathematical Form:

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ll.normal <- function(par, X, Y) {  
  beta <- par[1:ncol(X)]  
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- Calling it:

```
ll.normal(c(2,1,2,1,33,4,3.2),x,y)  
ll.normal(c(2,1,2,1,33,4,3.7),x,y)  
ll.normal(c(2,1,2,1,33,4,3.5),x,y)
```

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- Repeat Steps 1–3 $M = 1,000$ times, and plot a histogram of the results.

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4. Add fundamental uncertainty: draw $\tilde{y}_{i,2012} \sim N(\tilde{\mu}_{i,2012}, \tilde{\sigma}^2)$

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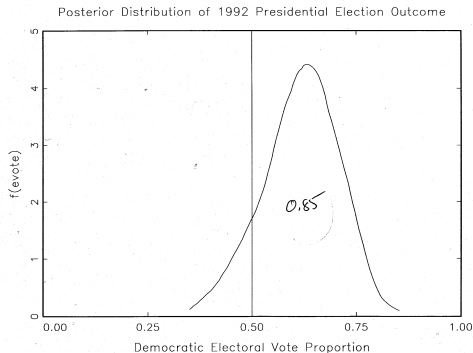
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 - Or, in our preferred notation, draw $\tilde{y}_{i,2012}$ from $N(X_{i,2012}\tilde{\beta}, \tilde{\sigma}^2)$

Actual Results (calculated before the election) for 1992

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4. Y_{it} and $Y_{i't'}$ are independent $\forall i \neq i'$ and $t \neq t'$, conditional on X and Z .

The log-likelihood:

Variance Function Models

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Variance Function Models

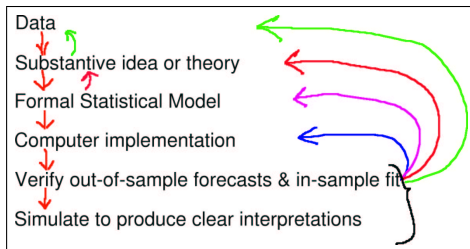
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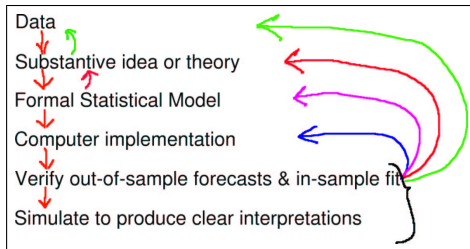
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- For what applications would this model be informative?

An Outline of the Research Process

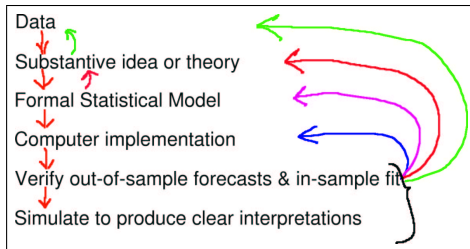


An Outline of the Research Process



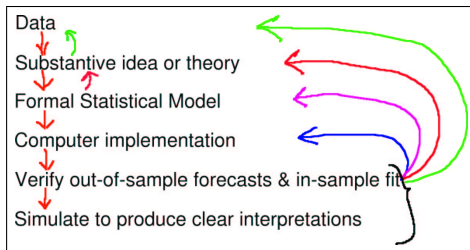
1. These figures are always wild simplifications.

An Outline of the Research Process



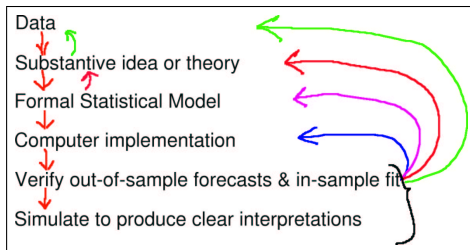
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An Outline of the Research Process



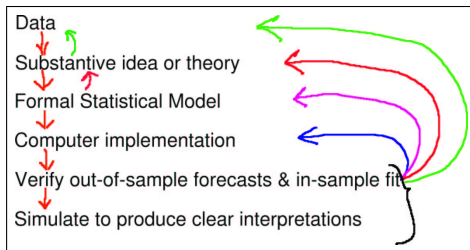
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An Outline of the Research Process



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4. A formal theory is often a useful addition, but not always necessary.

An Outline of the Research Process



1. These figures are always wild simplifications.
2. Items are roughly in order.
3. You can start at any point.
4. A formal theory is often a useful addition, but not always necessary.
5. Don't miss any parts.