Advanced Quantitative Research Methodology, Lecture Notes: Theories of Inference[1]

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The Problem of Inference

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3. A more reasonable, limited goal. Let $M = \{M^*, \theta\}$, where M^* is assumed & θ is to be estimated:

$$P(\theta|y, M^*) \equiv P(\theta|y)$$

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6. $L(\theta|y)$ is a function: for y fixed at the observed values, it gives the "likelihood" of any value of θ.

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9. Comparing values of $L(\theta|y)$ across data sets is meaningless. (just as you can't compare R^2 values across equations with different dependent variables.)
10. The likelihood principle: the data only affect inferences through the likelihood function

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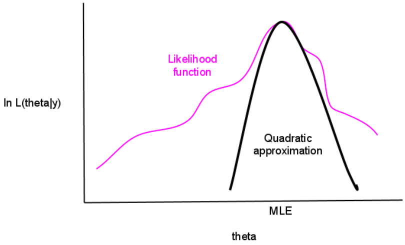

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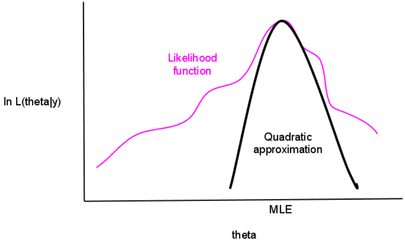

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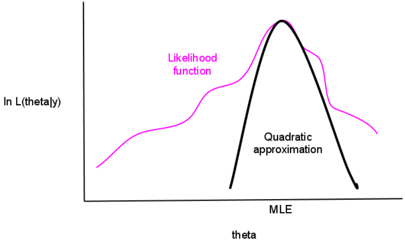

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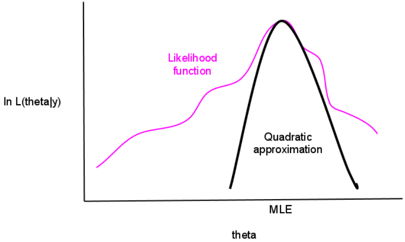

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- Uncertainty of point estimate: curvature at the maximum

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- $P(\theta)$, the prior density — the way Bayes differs from likelihood

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4. A philosophical assumption that nonsample information should matter (as it always does) *and* be formalized and included in all inferences.

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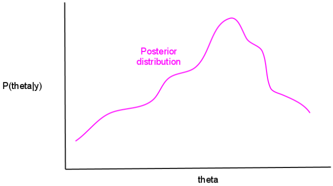
The posterior density, $P(\theta|y)$



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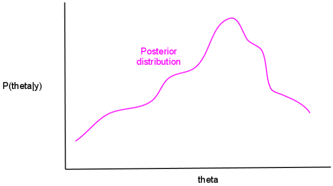

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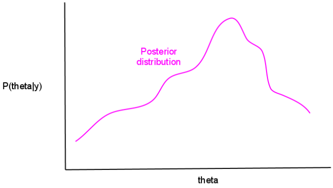

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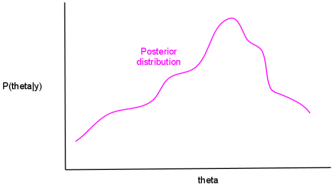

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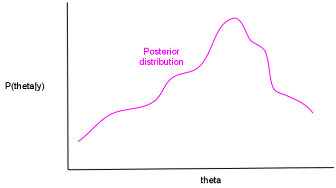

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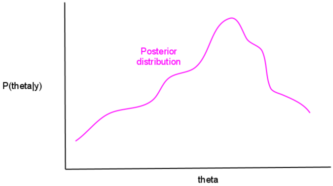

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- Bayesian inference obeys the likelihood principle: the data set only affects inferences through the likelihood function
- If $P(\theta) = 1$, i.e., is uniform in the relevant region, then $L(\theta|y) = P(\theta|y)$.

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- Few fights now between Bayesians and likelihoodists

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- (Type II error, the power to detect H_1 if it is true, is a consequence of choosing an estimator, not an ex ante decision like choosing α.)

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2. Strict but arbitrary distinction: null H_0 vs alternative H_1 hypotheses
3. All tests are "under" (i.e., assuming) H_0

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- H_0: $\beta = 0$ vs. H_1: $\beta > 0$
- Choose Type I error, probability of deciding H_1 is right when H_0 is really true: say $\alpha = 0.05$
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- or

$$(TS)_\beta|(\beta = 0) \equiv \frac{b - \beta}{\hat{\sigma}_b} \equiv \frac{b}{\hat{\sigma}_b} \sim N(0, 1).$$

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And then first collect your data. You may not revise your hypothesis or your theory.

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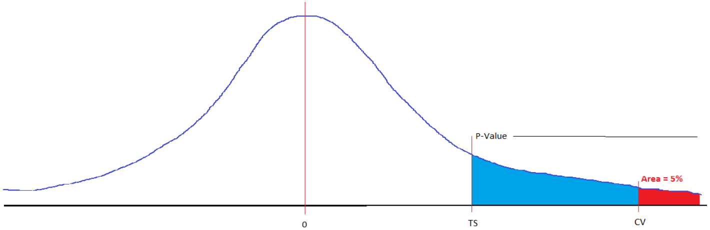

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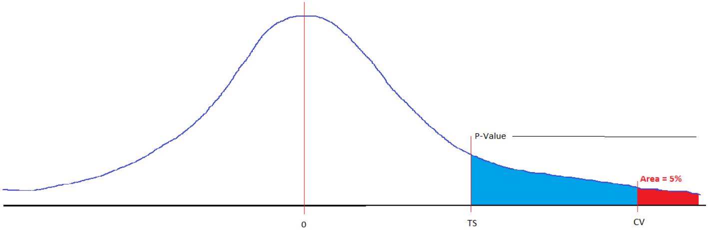

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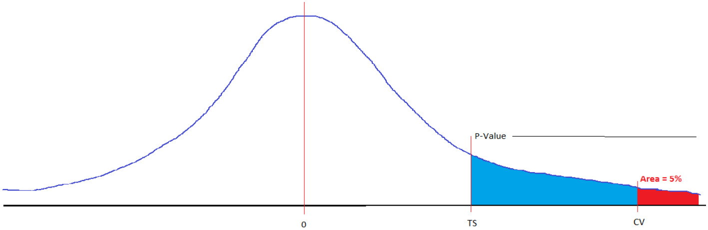

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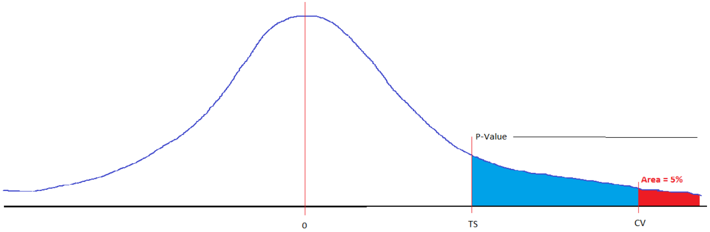

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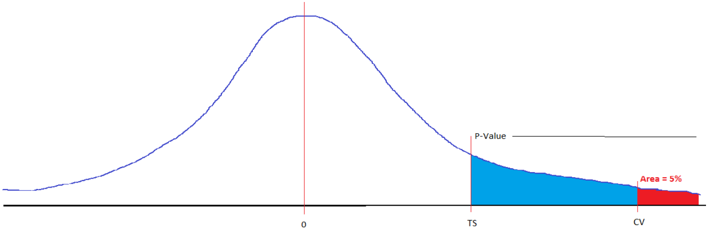

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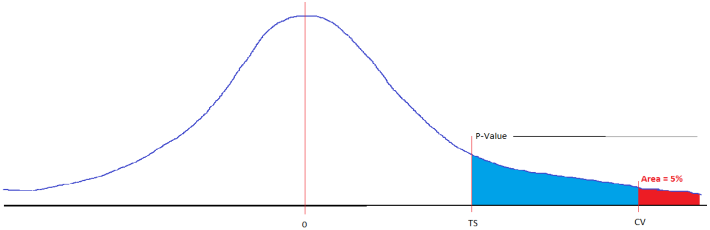

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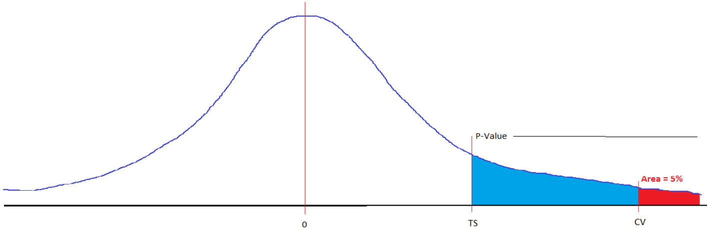

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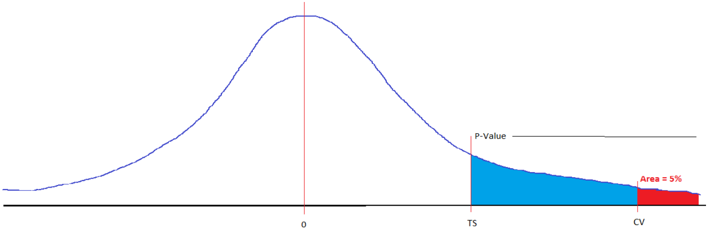

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- We can use likelihood to compute hypothesis tests and p-values.

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4. Methods for applied researchers: either useful or irrelevant

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- The key: No assumptions can be trusted; all theories of inference condition on assumptions and so data analysts always struggle trying to understand and get around them

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- What can you do with this probability density?

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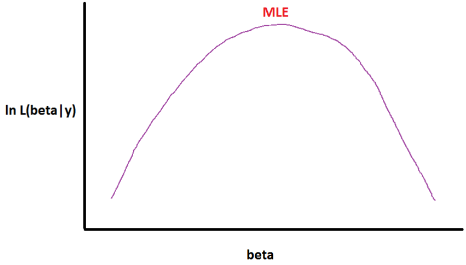
Log-likelihood interpretation



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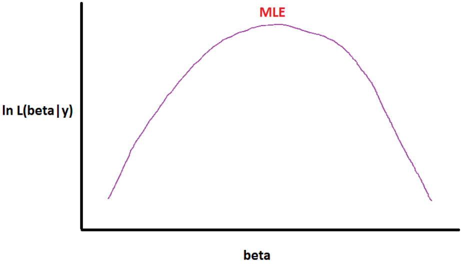

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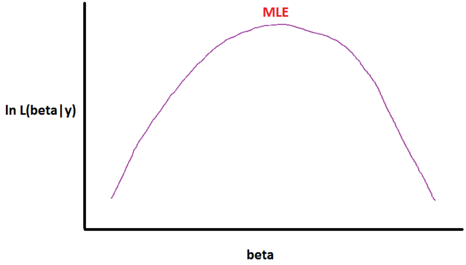

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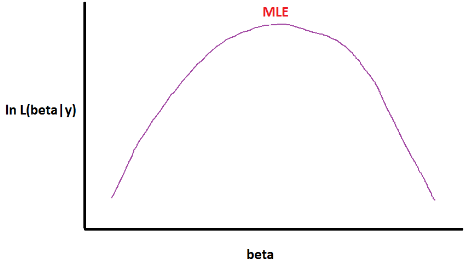

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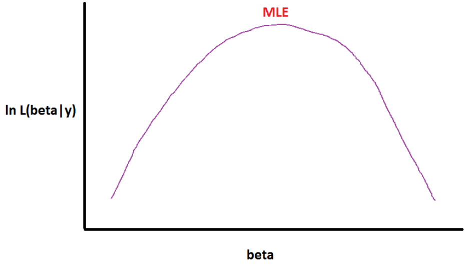

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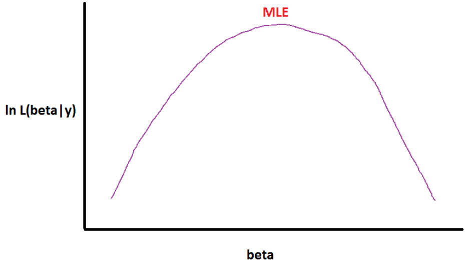

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5. No reason to summarize this curve with only the MLE

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- The problem of Flatland

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- Graphs

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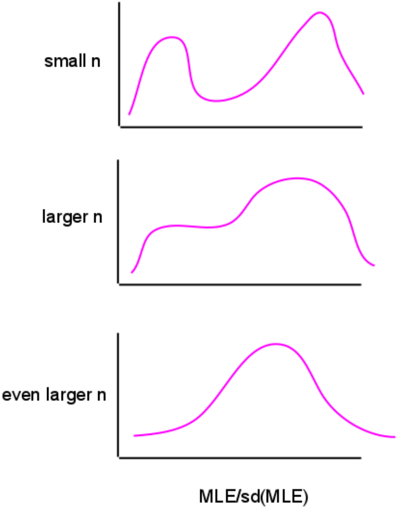
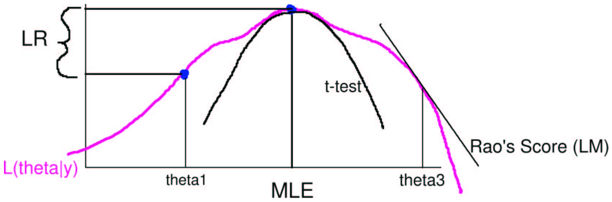
3. Asymptotic efficiency. The MLE contains as much information as can be packed into a point estimator.

Sampling distributions of the MLE: CLT vs LLN



Uncertainty: Likelihood Ratios for nested models



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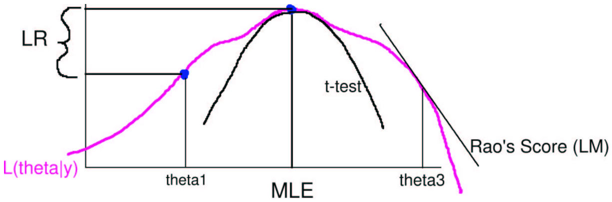

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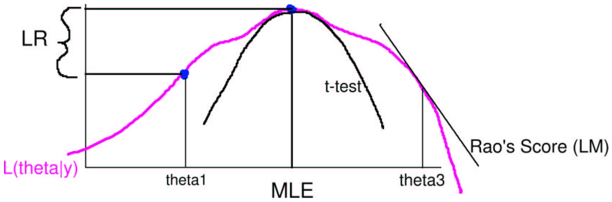

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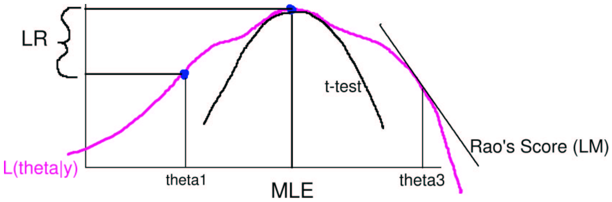
Uncertainty: Likelihood Ratios for nested models



- L^* is the likelihood value for the unrestricted model
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- $\implies L^* \geq L^*_R \implies \frac{L^*_R}{L^*} \leq 1$

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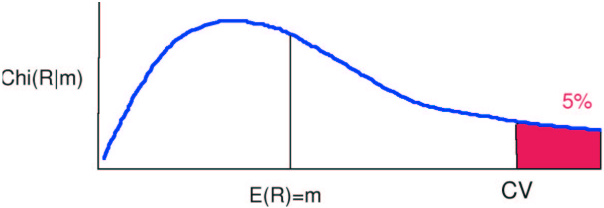

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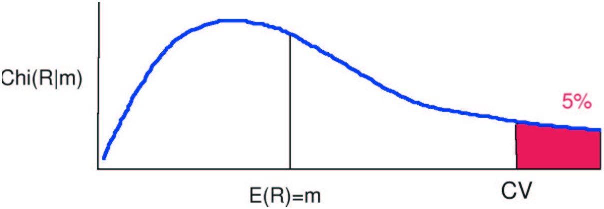

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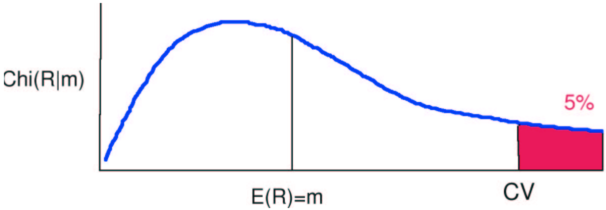

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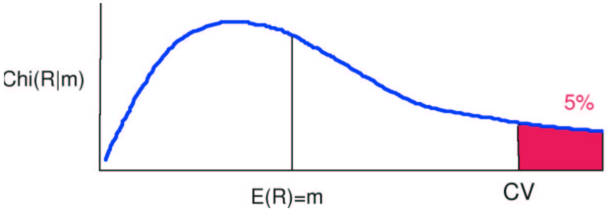

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Standard Errors

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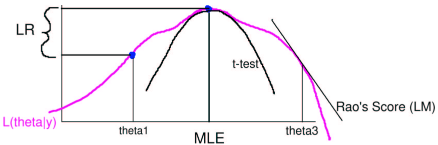

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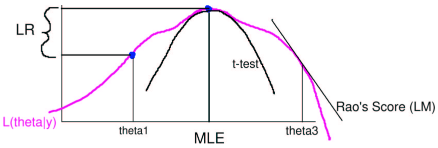

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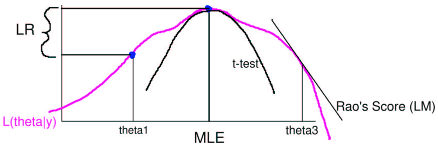

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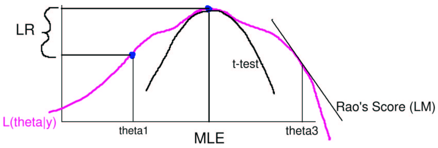

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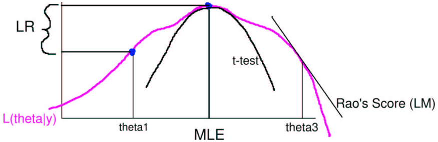

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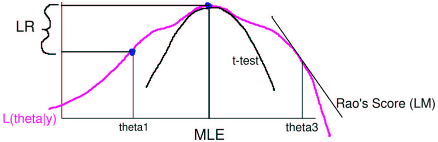

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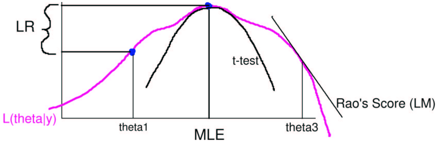

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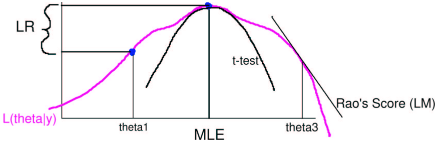

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   - the larger the likelihood ratio would be in comparing the MLE with *any* other parameter value.

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   - Statistical interpretation: variance and covariance across repeated samples

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E_i	The number of electoral college votes for each state in 2012

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R Code for the Log-Likelihood

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- Mathematical Form:

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- An R function:

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ll.normal <- function(par, X, Y) {
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- Calling it:

```
ll.normal(c(2,1,2,1,33,4,3.2),x,y)
ll.normal(c(2,1,2,1,33,4,3.7),x,y)
ll.normal(c(2,1,2,1,33,4,3.5),x,y)
```

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- Repeat Steps 1–3 $M = 1,000$ times, and plot a histogram of the results.

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4. Add fundamental uncertainty: draw $\tilde{y}_{i,2012} \sim N(y_{i,2012}|\tilde{\mu}_{i,2012}, \tilde{\sigma}^2)$

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Actual Results (calculated before the election) for 1992

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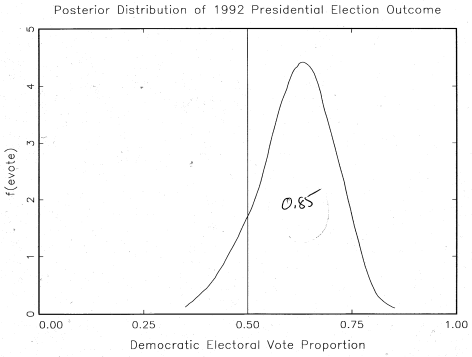

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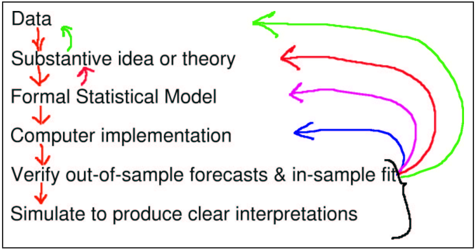
- For what applications would this model be informative?

An Outline of the Research Process



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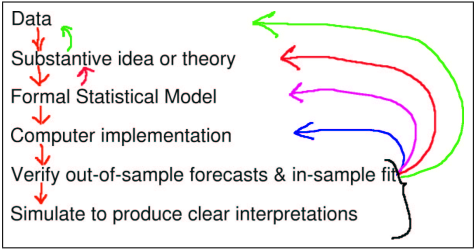

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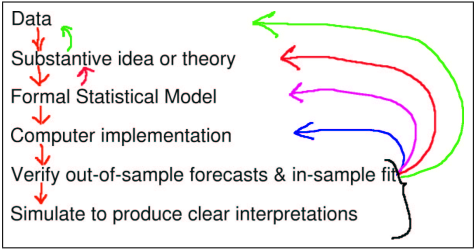

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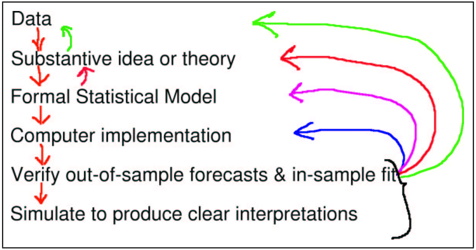

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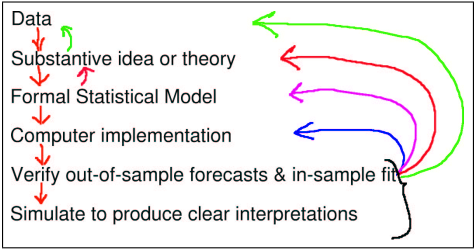

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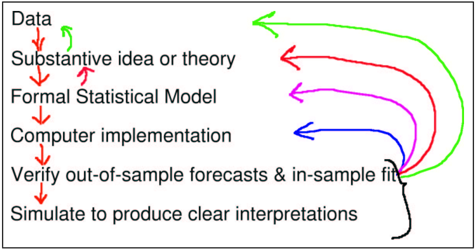

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