

Advanced Quantitative Research Methodology, Lecture Notes: Theories of Inference¹

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The Problem of Inference

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3. Define: $M = \{M^*, \theta\}$, where M^* is assumed and θ is to be estimated
4. More limited goal: $P(\theta|y, M^*) \equiv P(\theta|y)$.

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- 9. The two differ on the rest

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6. $L(\theta|y)$ is a function: for y fixed at the observed values, it gives the “likelihood” of any value of θ .

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10. The **likelihood principle**: the data only affect inferences through the likelihood function

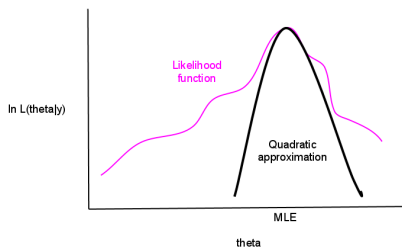
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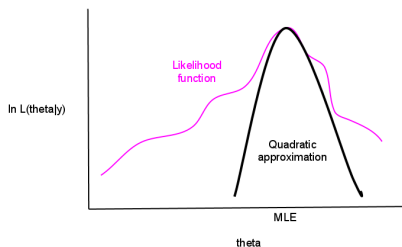
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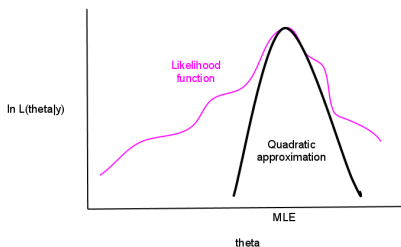
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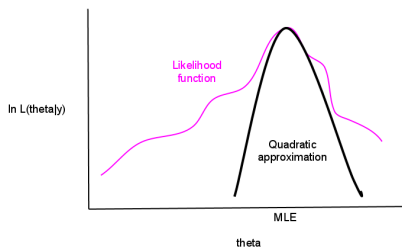
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- Uncertainty of point estimate: **curvature at the maximum**

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- $P(\theta)$, the prior density — the way Bayes differs from likelihood

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3. An annoyance: the “other information” is required
4. A philosophical assumption that nonsample information should matter (as it always does) *and* be formalized and included in all inferences.

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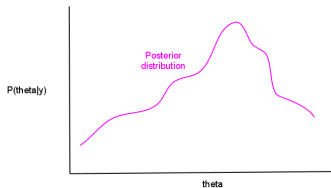
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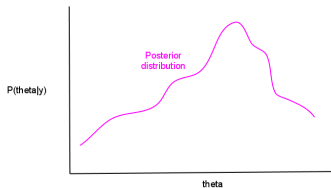
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4. If variables A and B are both unknown, then the distribution of A alone is $P(A) = \int P(A, B)dB = \int P(A|B)P(B)dB$.

The posterior density, $P(\theta|y)$

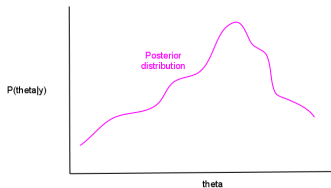


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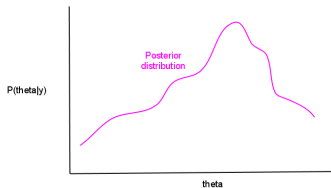
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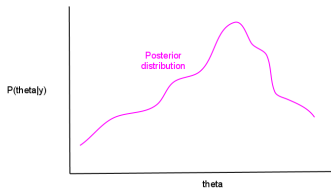
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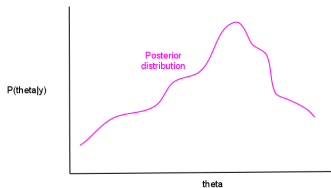
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- **Bayesian inference** obeys the **likelihood principle**: the data set only affects inferences through the likelihood function
- If $P(\theta) = 1$, i.e., is uniform in the relevant region, then $L(\theta|y) = P(\theta|y)$.

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- Few fights now between Bayesians and likelihoodists

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 - (c) Type II error, the power to detect H_1 if it is true, is a consequence of choosing an estimator, not an ex ante decision like choosing α .
 - (d) Assume n is large enough for the CLT to kick in

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 - (a) $H_0: \beta = 0$ vs. $H_1: \beta > 0$
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 - (c) Type II error, the power to detect H_1 if it is true, is a consequence of choosing an estimator, not an ex ante decision like choosing α .
 - (d) Assume n is large enough for the CLT to kick in
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A 3rd Theory: Neyman-Pearson Hypothesis Testing

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 - (f) or

$$(TS)_{\beta}|(\beta = 0) \equiv \frac{b - \beta}{\hat{\sigma}_b} \equiv \frac{b}{\hat{\sigma}_b} \sim N(0, 1).$$

Neyman-Pearson Hypothesis Testing

Neyman-Pearson Hypothesis Testing

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Neyman-Pearson Hypothesis Testing

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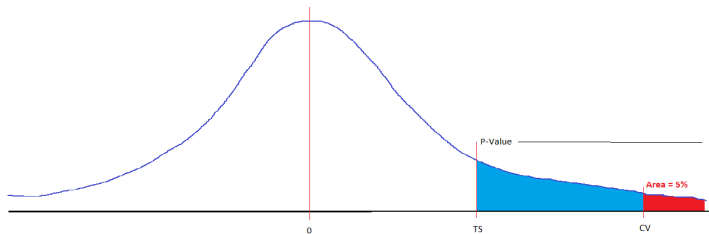
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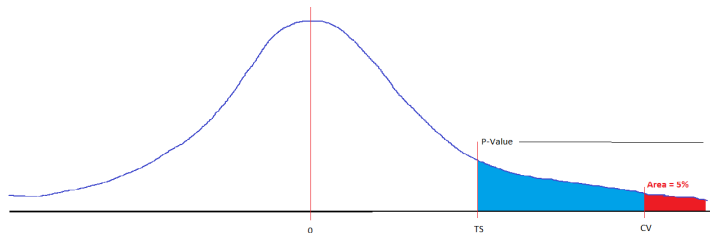
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And then first collect your data. You may not revise your hypothesis or your theory.

Neyman-Pearson Hypothesis Testing

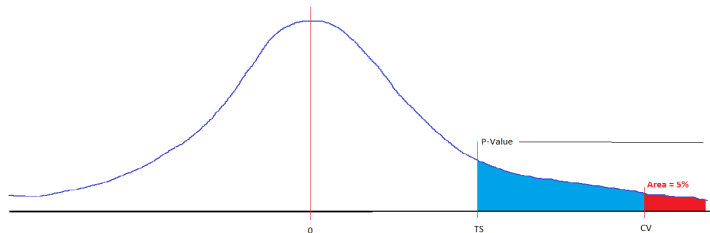


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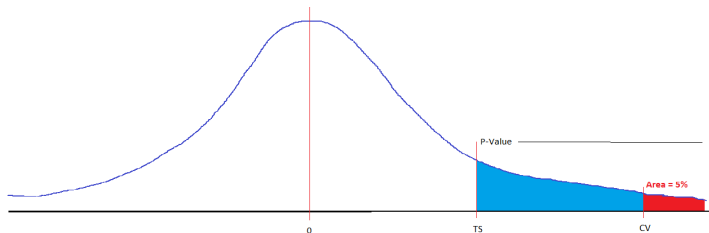
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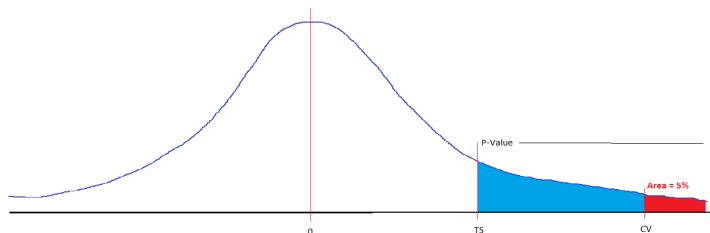
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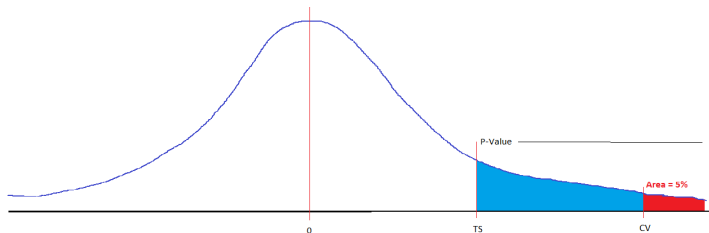
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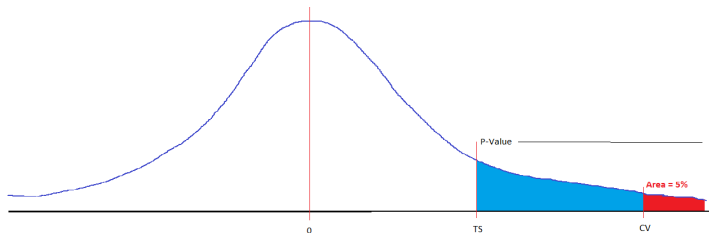
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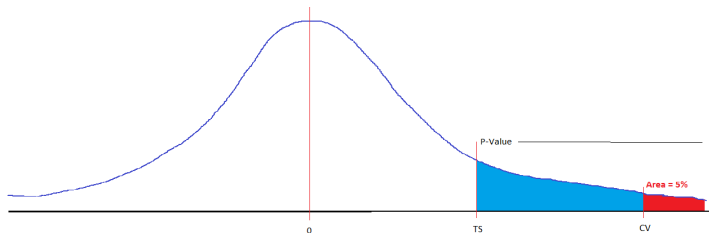
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 8. We can use likelihood to compute hypothesis tests and p -values.

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3. The right theory of inference: **utilitarianism**

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- The key: No assumptions can be trusted; all theories of inference condition on assumptions and so data analysts always struggle trying to understand and get around them

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- What can you do with this density?

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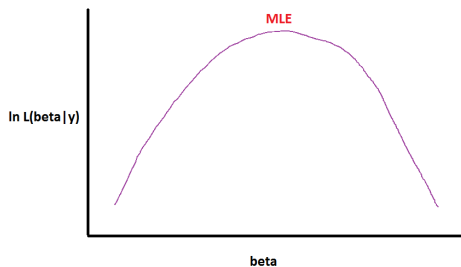
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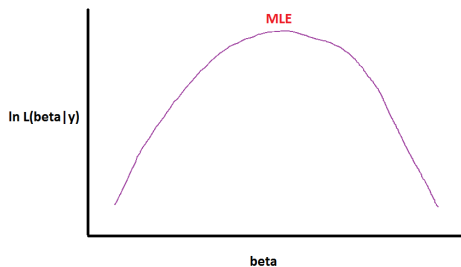
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Log-likelihood interpretation

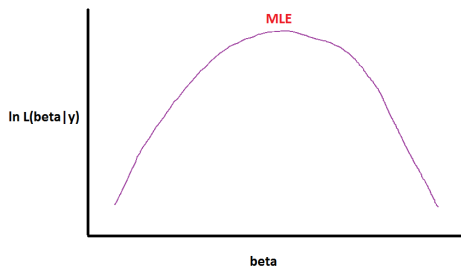


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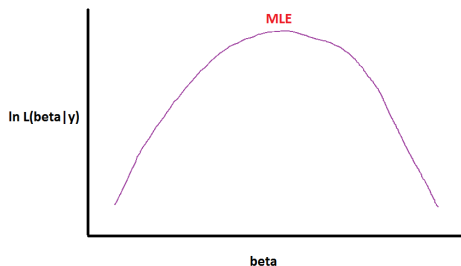
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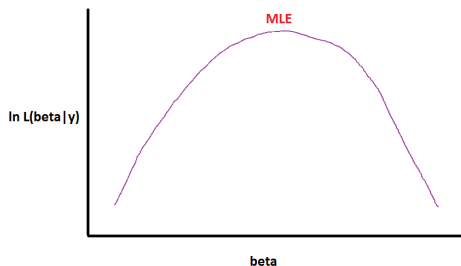
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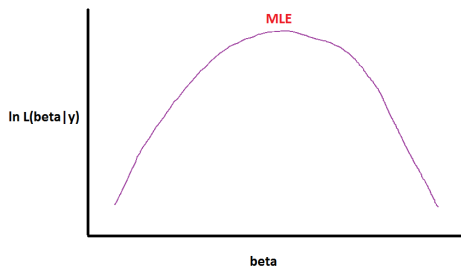
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- The problem of Flatland

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- Graphs

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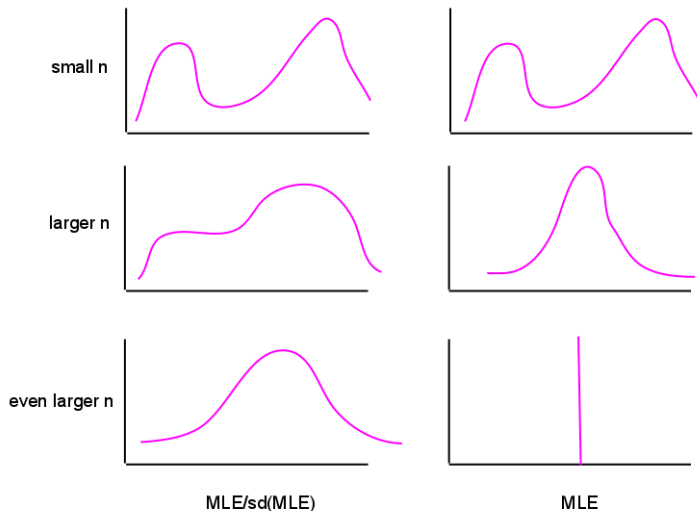
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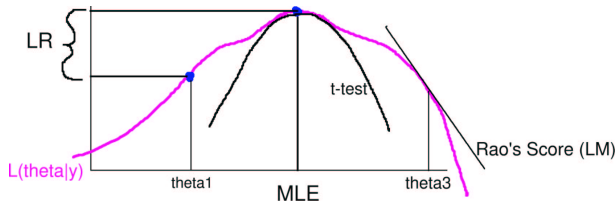
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- 3 **Asymptotic efficiency**. The MLE contains as much information as can be packed into a point estimator.

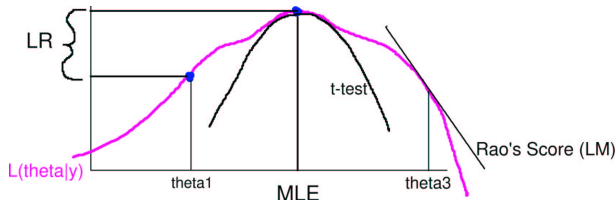
Sampling distributions of the MLE: CLT vs LLN



Uncertainty: Likelihood Ratios for nested models

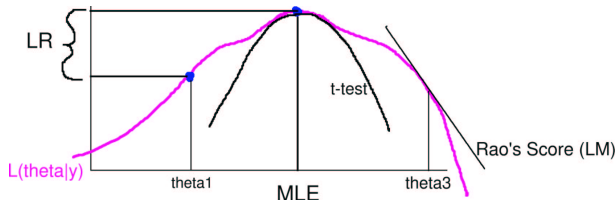


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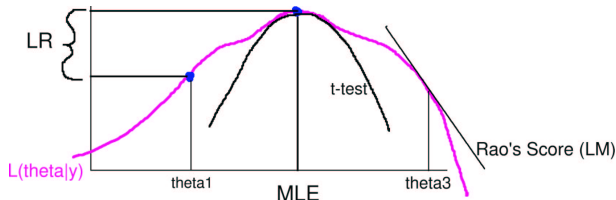
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- $\implies L^* \geq L_R^* \implies \frac{L_R^*}{L^*} \leq 1$

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Interpreted as a *risk ratio*.

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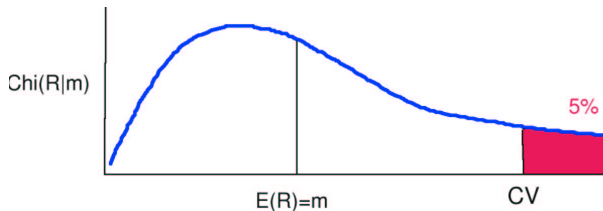
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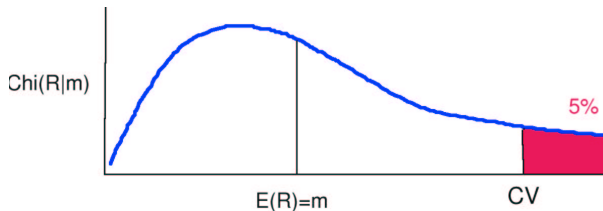
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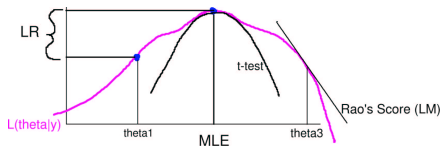
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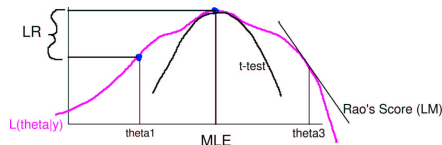
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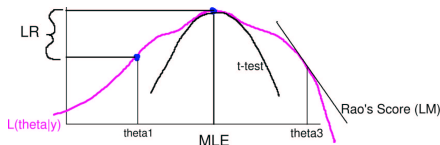


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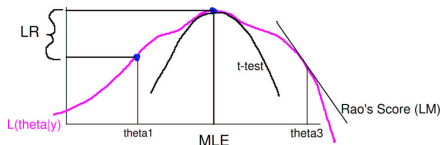
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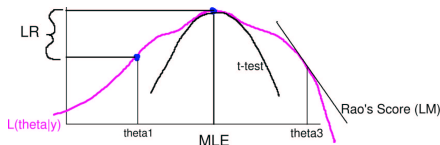
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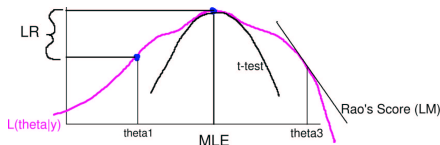
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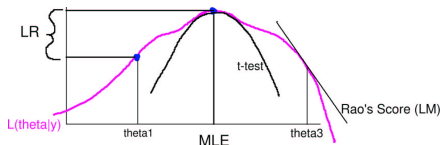
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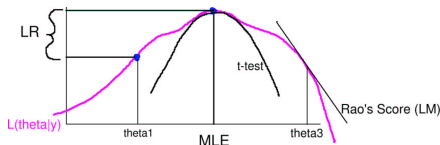
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E_i	The number of electoral college votes for each state in 2012

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$$\begin{aligned} L(\mu_{it}, \sigma | y_{it}) &\propto N(y_{it} | \mu_{it}, \sigma^2) \\ &= (2\pi\sigma^2)^{-1/2} e^{\frac{-(y_{it} - \mu_{it})^2}{2\sigma^2}} \end{aligned}$$

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- Compute and save $\hat{V}(\hat{\theta})$, which is $k + 2 \times k + 2$

R Code for the Log-Likelihood

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- Mathematical Form:

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- The function:

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ll.normal <- function(par, X, Y) {  
  beta <- par[1:ncol(X)]  
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- Calling it:

```
ll.normal(c(2,1,2,1,33,4,\alert{3.2}),x,y)  
ll.normal(c(2,1,2,1,33,4,\alert{3.7}),x,y)  
ll.normal(c(2,1,2,1,33,4,\alert{3.5}),x,y)
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- Repeat Steps 1–3 $M = 1,000$ times, and plot a histogram of the results.

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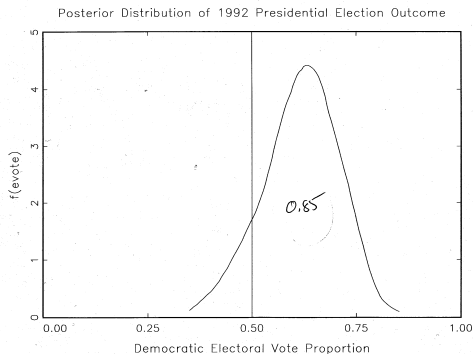
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 - Or, in our preferred notation, draw $\tilde{y}_{i,2012}$ from $N(X_{i,2012}\tilde{\beta}, \tilde{\sigma}^2)$

Actual Results (calculated before the election) for 1992

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1. $Y_{it} \sim N(y_{it} | \mu_{it}, \sigma_{it}^2)$
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3. $\sigma_{it}^2 = \exp(z_{it}\gamma)$, where z_{it} is a vector of explanatory variables possibly overlapping x_{it}
4. Y_{it} and $Y_{i't'}$ are independent $\forall i \neq i'$ and $t \neq t'$, conditional on X and Z .

The log-likelihood:

$$\ln L(\beta, \sigma^2 | y) = \sum_{i=1}^n \sum_{t=1}^T -\frac{1}{2} \left[\ln \sigma^2 + \frac{(y_{it} - X_{it}\beta)^2}{\sigma^2} \right]$$

Variance Function Models

1. $Y_{it} \sim N(y_{it}|\mu_{it}, \sigma_{it}^2)$
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The log-likelihood:

$$\begin{aligned}\ln L(\beta, \sigma^2|y) &= \sum_{i=1}^n \sum_{t=1}^T -\frac{1}{2} \left[\ln \sigma^2 + \frac{(y_{it} - X_{it}\beta)^2}{\sigma^2} \right] \\ &= \sum_{i=1}^n \sum_{t=1}^T -\frac{1}{2} \left[z_{it}\gamma + \frac{(y_{it} - X_{it}\beta)^2}{\exp(z_{it}\gamma)} \right]\end{aligned}$$

Variance Function Models

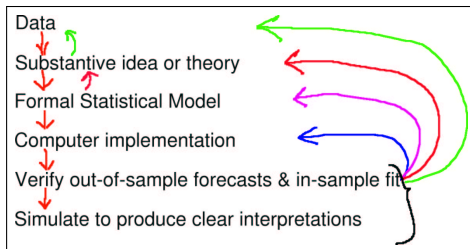
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The log-likelihood:

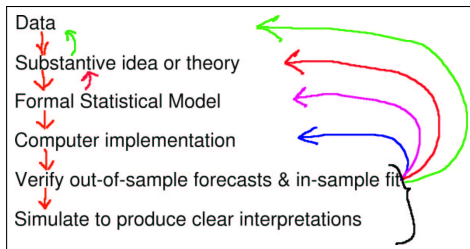
$$\begin{aligned}\ln L(\beta, \sigma^2|y) &= \sum_{i=1}^n \sum_{t=1}^T -\frac{1}{2} \left[\ln \sigma^2 + \frac{(y_{it} - X_{it}\beta)^2}{\sigma^2} \right] \\ &= \sum_{i=1}^n \sum_{t=1}^T -\frac{1}{2} \left[z_{it}\gamma + \frac{(y_{it} - X_{it}\beta)^2}{\exp(z_{it}\gamma)} \right]\end{aligned}$$

- For what applications would this model be informative?

An Outline of the Research Process

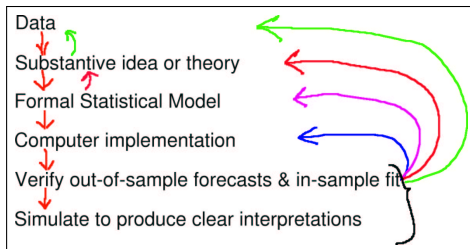


An Outline of the Research Process



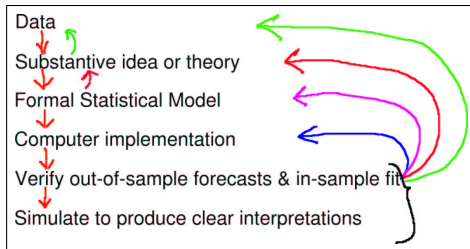
1. These figures are always wild simplifications.

An Outline of the Research Process



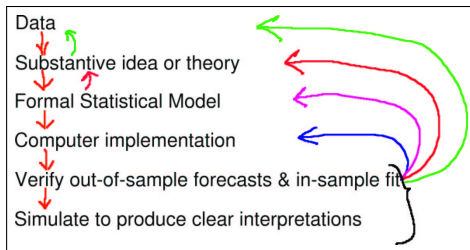
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An Outline of the Research Process



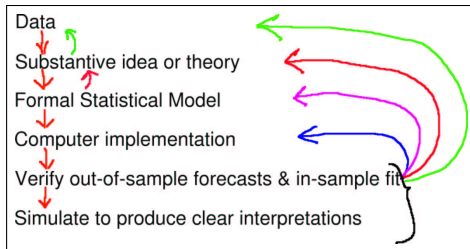
1. These figures are always wild simplifications.
2. Items are roughly in order.
3. You can start at any point.

An Outline of the Research Process



1. These figures are always wild simplifications.
2. Items are roughly in order.
3. You can start at any point.
4. A formal theory is often a useful addition, but not always necessary.

An Outline of the Research Process



1. These figures are always wild simplifications.
2. Items are roughly in order.
3. You can start at any point.
4. A formal theory is often a useful addition, but not always necessary.
5. Don't miss any parts.