Advanced Quantitative Research Methodology, Lecture Notes: Theories of Inference[1]

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3. Define: $M = \{M^*, \theta\}$, where M^* is assumed and θ is to be estimated
4. More limited goal: $\text{P}(\theta|y, M^*) \equiv \text{P}(\theta|y)$.

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6. $L(\theta|y)$ is a function: for y fixed at the observed values, it gives the "likelihood" of any value of θ.

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10. The likelihood principle: the data only affect inferences through the likelihood function

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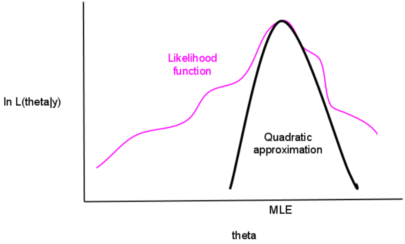

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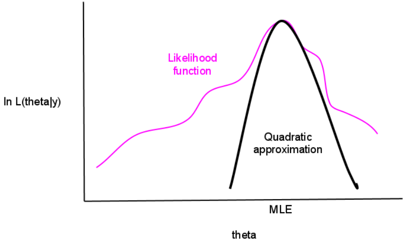

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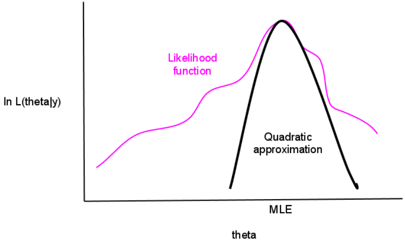

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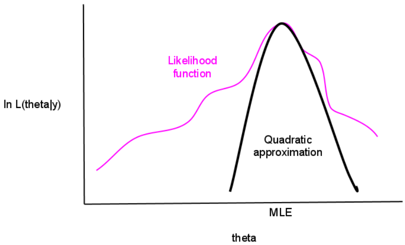

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- Uncertainty of point estimate: curvature at the maximum

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- $P(\theta)$, the prior density — the way Bayes differs from likelihood

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3. An annoyance: the "other information" is required
4. A philosophical assumption that nonsample information should matter (as it always does) *and* be formalized and included in all inferences.

Principles of Bayesian analysis

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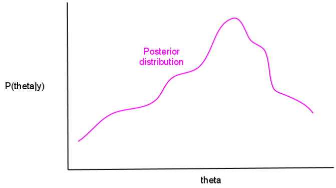
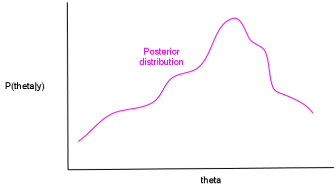
The posterior density, $P(\theta|y)$





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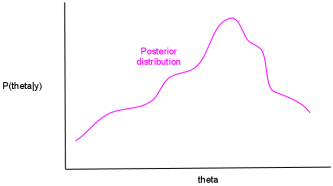

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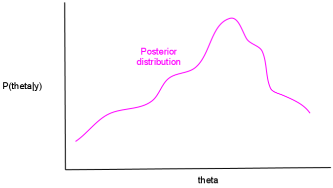

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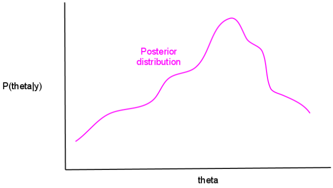

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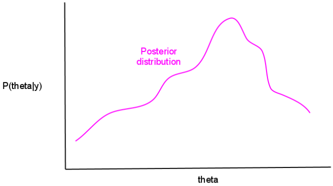

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- If $P(\theta) = 1$, i.e., is uniform in the relevant region, then $L(\theta|y) = P(\theta|y)$.

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- Few fights now between Bayesians and likelihoodists

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4. For example, is $\beta = 0$ in $E(Y) = \beta_0 + \beta X$?
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   (b) Choose Type I error, probability of deciding H_1 is right when H_0 is really true: say $\alpha = 0.05$

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   (f) or

$$(TS)_\beta|(\beta = 0) \equiv \frac{b - \beta}{\hat{\sigma}_b} \equiv \frac{b}{\hat{\sigma}_b} \sim N(0, 1).$$

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And then first collect your data. You may not revise your hypothesis or your theory.

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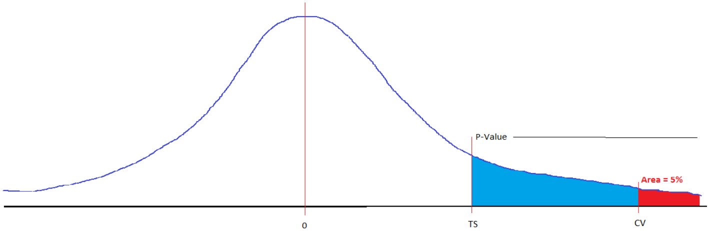

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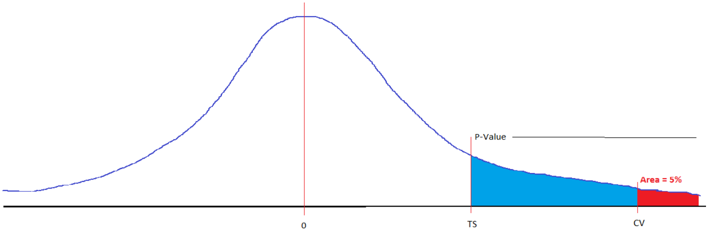

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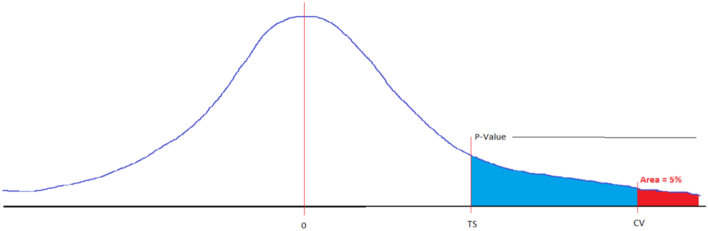

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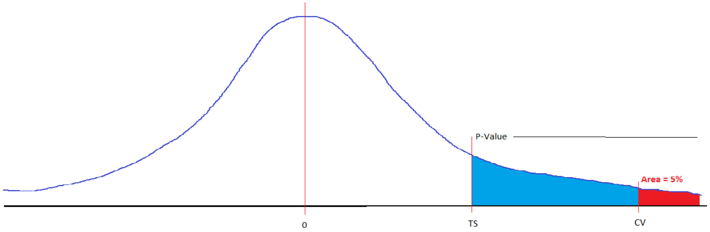

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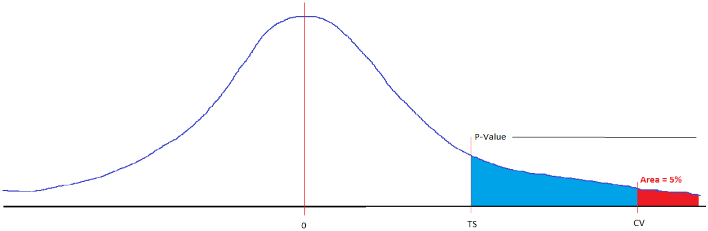

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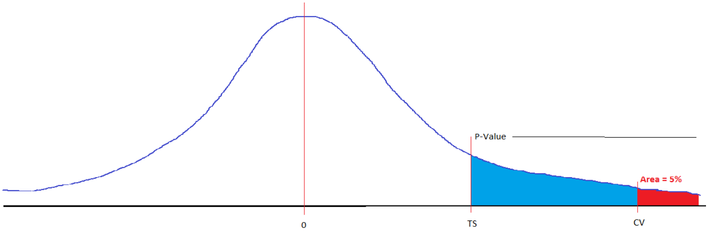

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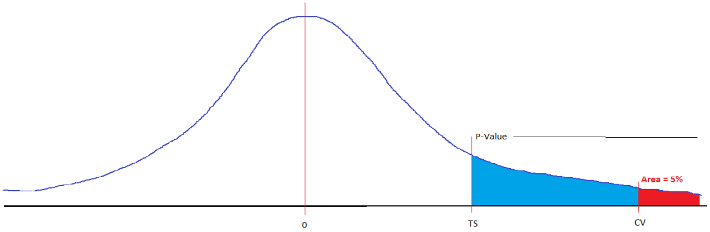

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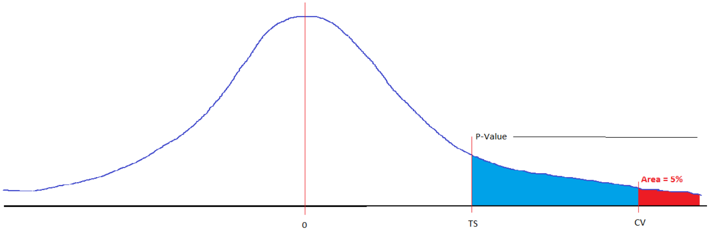

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8. We can use likelihood to compute hypothesis tests and p-values.

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- The key: No assumptions can be trusted; all theories of inference condition on assumptions and so data analysts always struggle trying to understand and get around them

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- What can you do with this density?

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$$L(\beta|y) = k(y) \prod_{i=1}^{n} f_{\text{stn}}(y_i|\beta)$$

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&= \ln[k(y)] + \sum_{i=1}^{n}\ln[(2\pi)^{-1/2}] - \sum_{i=1}^{n}\frac{1}{2}(y_i-\beta)^2 \\
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Stylized Normal Likelihood Function

$$
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L(\beta|y) &= k(y)\prod_{i=1}^{n} f_{\text{stn}}(y_i|\beta) \\
&\propto \prod_{i=1}^{n} f_{\text{stn}}(y_i|\beta) \\
&= \prod_{i=1}^{n}(2\pi)^{-1/2}\exp\left(\frac{-(y_i-\beta)^2}{2}\right)
\end{aligned}
$$

the relative likelihood of β (conditional on the model) having generated the data we observe.
The log-likelihood (Recall: $\ln(ab) = \ln(a) + \ln(b)$):

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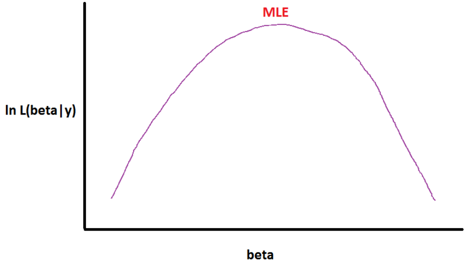

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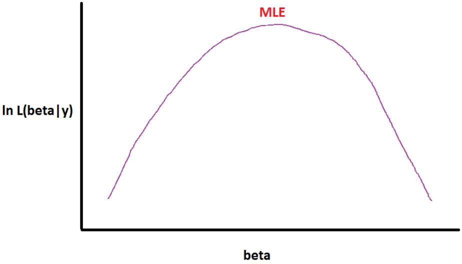

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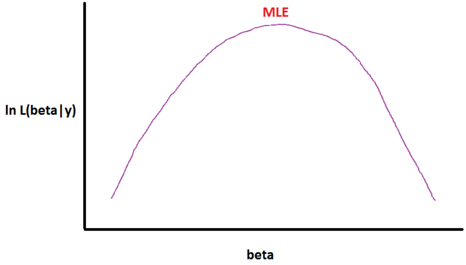

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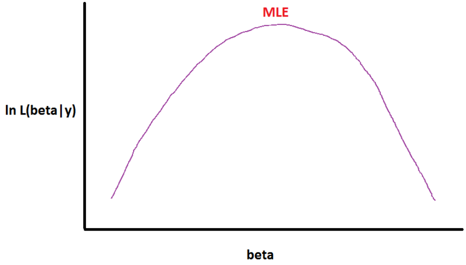

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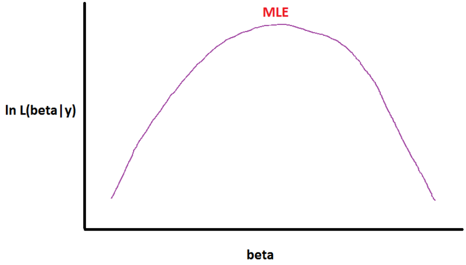

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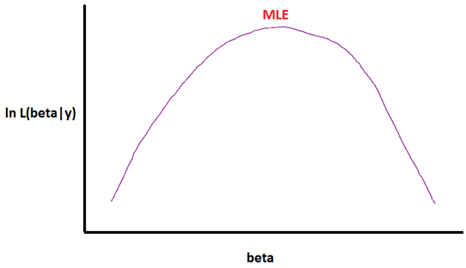

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5. No reason to summarize this curve with only the MLE

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- The problem of Flatland

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- Graphs

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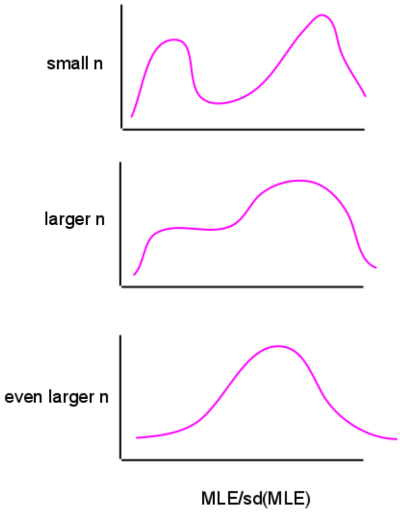
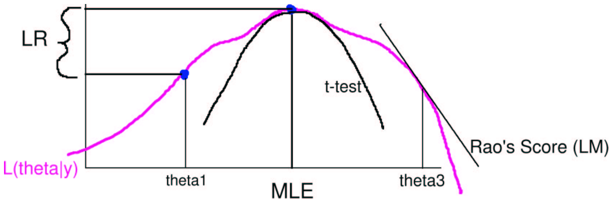
3. Asymptotic efficiency. The MLE contains as much information as can be packed into a point estimator.

Sampling distributions of the MLE: CLT vs LLN




LR
L(theta|y)
theta1
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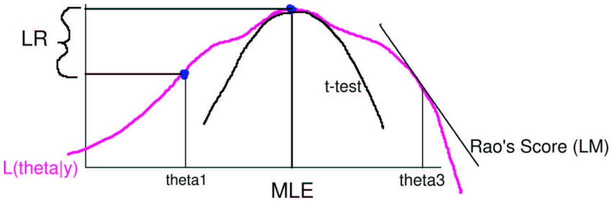
theta3
t-test
Rao's Score (LM)

Uncertainty: Likelihood Ratios for nested models



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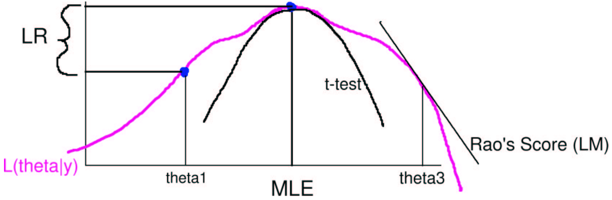

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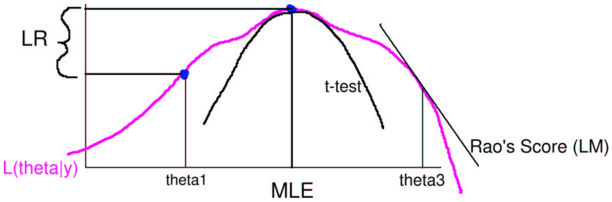

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- $\implies L^* \geq L^*_R \implies \frac{L^*_R}{L^*} \leq 1$

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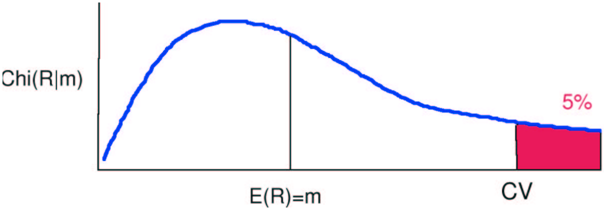

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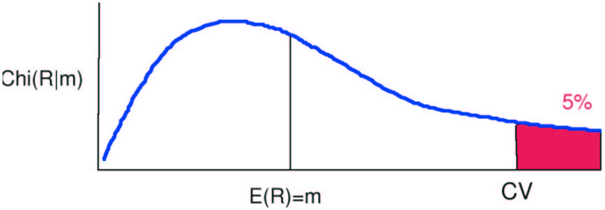

3. Disadvantage: Too many likelihood ratio tests may be required to test all points of interest

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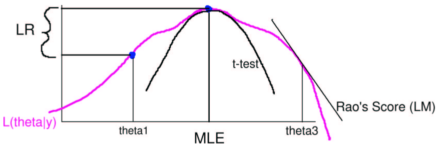

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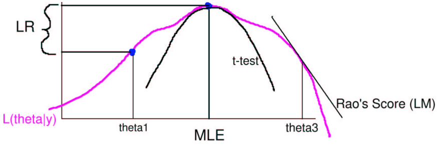

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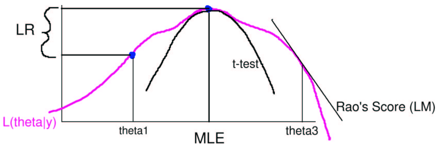

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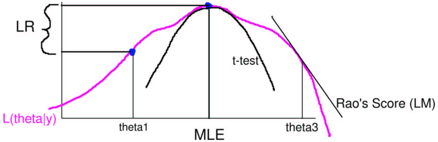

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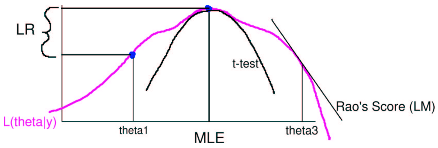

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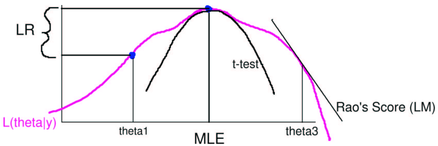

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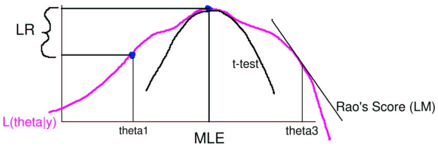

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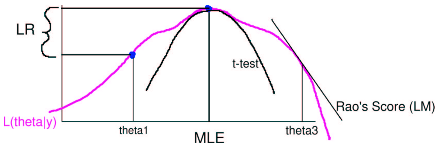

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E_i	The number of electoral college votes for each state in 2012

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$$\begin{aligned} L(\mu_{it}, \sigma | y_{it}) &\propto N(y_{it}|\mu_{it}, \sigma^2) \\ &= (2\pi\sigma^2)^{-1/2} e^{\frac{-(y_{it}-\mu_{it})^2}{2\sigma^2}} \end{aligned}$$

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- Compute and save $\hat{V}(\hat{\theta})$, which is $k + 2 \times k + 2$

R Code for the Log-Likelihood

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- Mathematical Form:

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ll.normal <- function(par, X, Y) {
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- Calling it:

```
ll.normal(c(2,1,2,1,33,4,3.2),x,y)
ll.normal(c(2,1,2,1,33,4,3.7),x,y)
ll.normal(c(2,1,2,1,33,4,3.5),x,y)
```

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Predictive distribution of electoral college delegates in 2012

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- Draw many simulations of $y_{i,2012}$ ($\tilde{y}_{i,2012}$) from its posterior distribution for U.S. state i, $\mathrm{P}(y_{i,2012}|y_{it}, X_{it})$, i.e. $\mathrm{P}(\texttt{quantity of interest}|\texttt{data})$. (Details shortly.)

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- Repeat Steps 1–3 $M = 1,000$ times, and plot a histogram of the results.

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- (How does this differ from allocating all E_i based on the point estimate?)

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4. Add fundamental uncertainty: draw $y_{i,2012}$ from $N(y_{i,2012}|\tilde{\mu}_{i,2012}, \tilde{\sigma}^2)$

How to do it with a LS Regression Program

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   - Or, in our preferred notation, draw $\tilde{y}_{i,2012}$ from $N(X_{i,2012}\tilde{\beta}, \tilde{\sigma}^2)$

Actual Results (calculated before the election) for 1992

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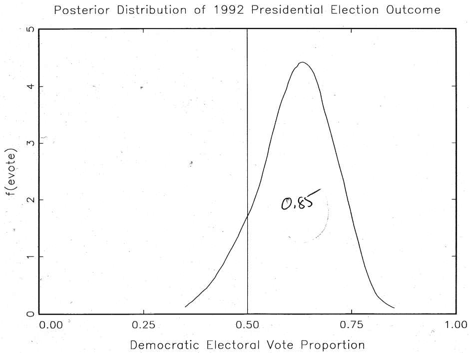

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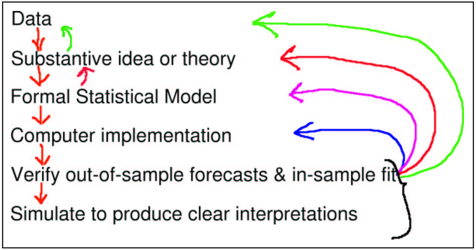
- For what applications would this model be informative?

An Outline of the Research Process



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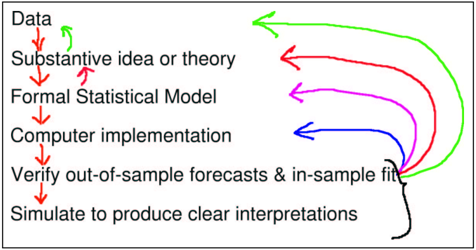

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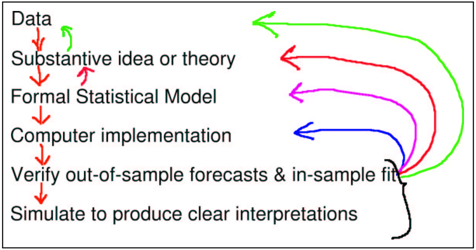

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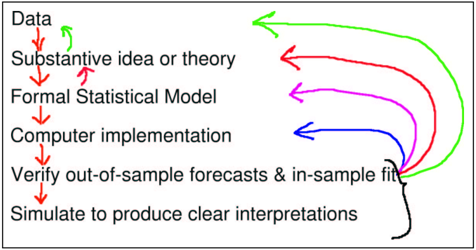

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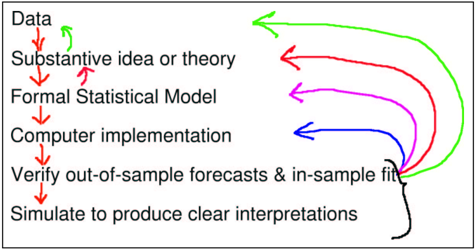

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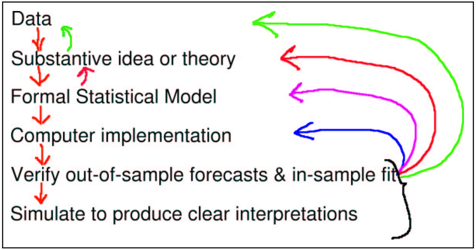

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