Advanced Quantitative Research Methodology, Lecture Notes: Theories of Inference[1]

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The Problem of Inference

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3. A more reasonable, limited goal. Let $M = \{M^*, \theta\}$, where M^* is assumed & θ is to be estimated:

$$P(\theta|y, M^*) \equiv P(\theta|y)$$

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8. The two differ on the rest

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6. $L(\theta|y)$ is a function: for y fixed at the observed values, it gives the "likelihood" of any value of θ.

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10. The likelihood principle: the data only affect inferences through the likelihood function

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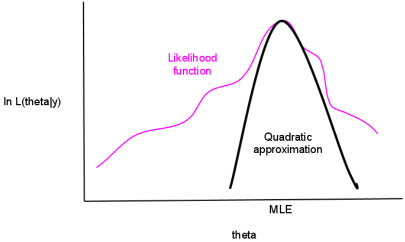

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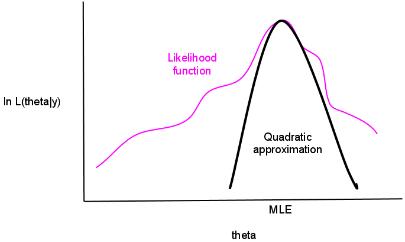

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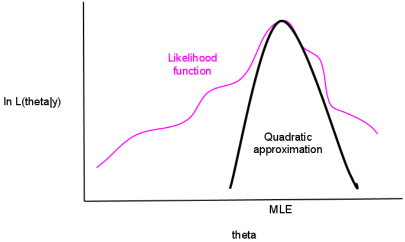

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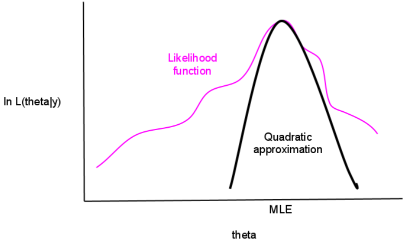

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- Uncertainty of point estimate: curvature at the maximum

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- $\mathrm{P}(\theta)$, the prior density — the way Bayes differs from likelihood

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3. An annoyance: the "other information" is required
4. A philosophical assumption that nonsample information should matter (as it always does) *and* be formalized and included in all inferences.

Principles of Bayesian analysis

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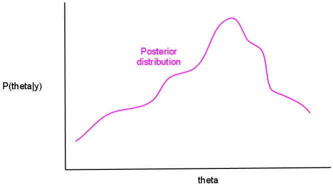
4. If variables A and B are both unknown, then the distribution of A alone is $P(A) = \int \mathrm{P}(A, B)dB = \int P(A|B)P(B)dB$.

The posterior density, $P(\theta|y)$



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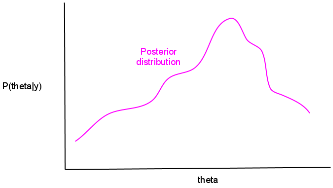

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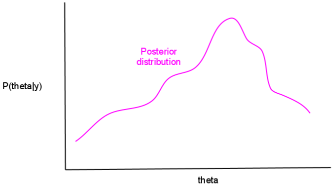

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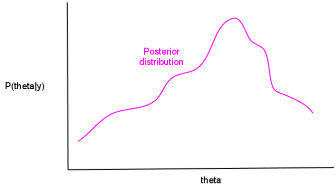

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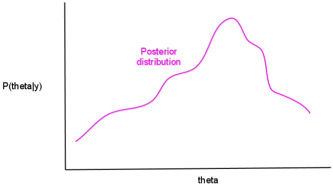

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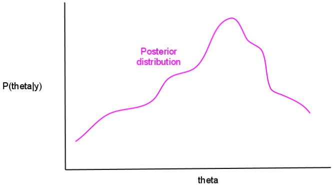

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- Bayesian inference obeys the likelihood principle: the data set only affects inferences through the likelihood function
- If $P(\theta) = 1$, i.e., is uniform in the relevant region, then $L(\theta|y) = P(\theta|y)$.

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- Few fights now between Bayesians and likelihoodists

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$$(TS)_\beta|(\beta = 0) \equiv \frac{b - \beta}{\hat{\sigma}_b} \equiv \frac{b}{\hat{\sigma}_b} \sim N(0, 1).$$

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  And then first collect your data. You may not revise your hypothesis or your theory.

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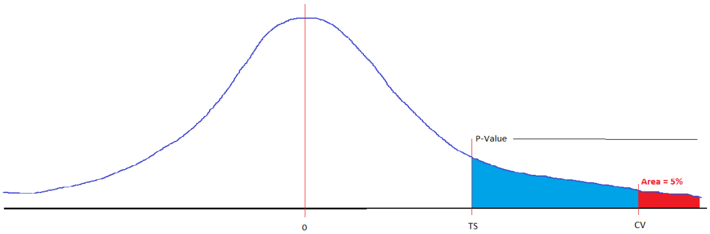

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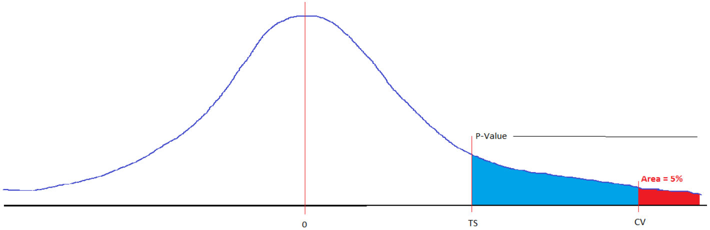

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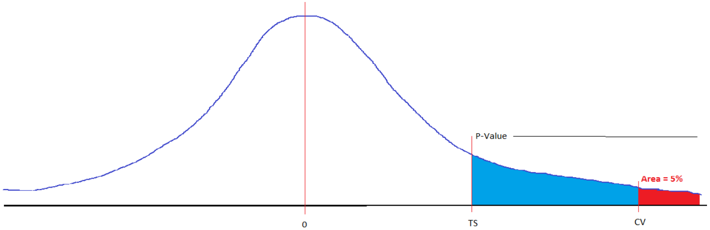

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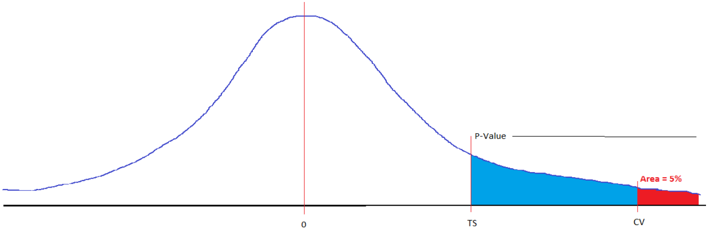

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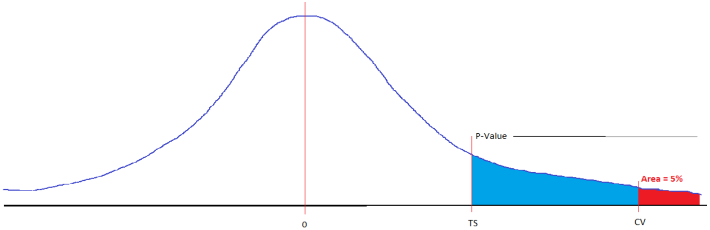

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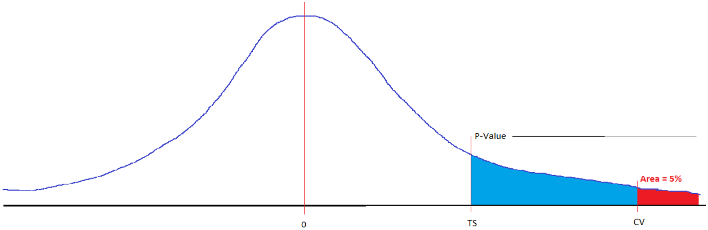

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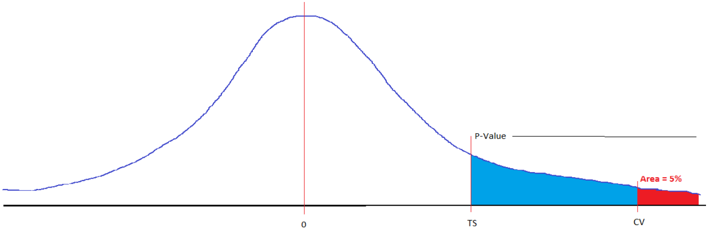

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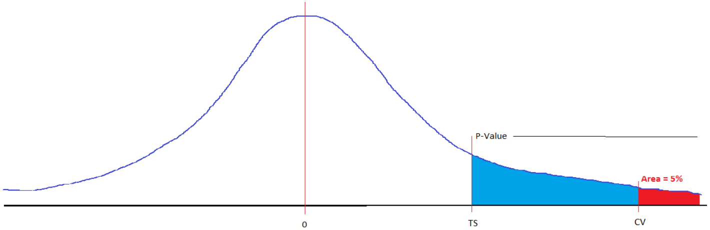

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- We can use likelihood to compute hypothesis tests and p-values.

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4. Methods for applied researchers: either useful or irrelevant

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- The key: No assumptions can be trusted; all theories of inference condition on assumptions and so data analysts always struggle trying to understand and get around them

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- What can you do with this probability density?

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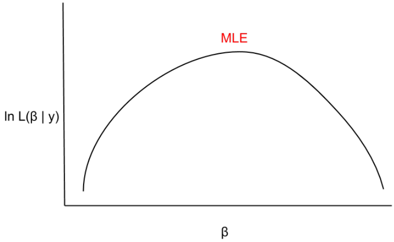
The log-likelihood (Recall: $\ln(ab) = \ln(a) + \ln(b)$):

$$\ln L(\beta|y) = \ln[k(y)] + \sum_{i=1}^{n} \ln f_{\text{stn}}(y_i|\beta)$$

$$= \ln[k(y)] + \sum_{i=1}^{n} \ln[(2\pi)^{-1/2}] - \sum_{i=1}^{n} \frac{1}{2}(y_i-\beta)^2$$

$$\doteq \sum_{i=1}^{n} -\frac{1}{2}(y_i-\beta)^2 = -\frac{1}{2}\sum_{i=1}^{n}(y_i-\beta)^2$$

Log-likelihood interpretation



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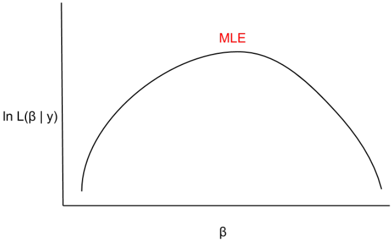

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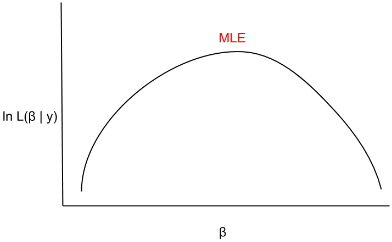

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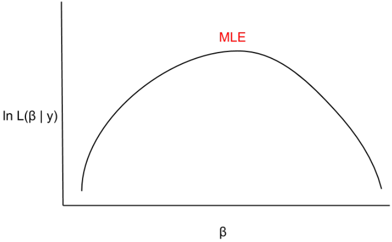

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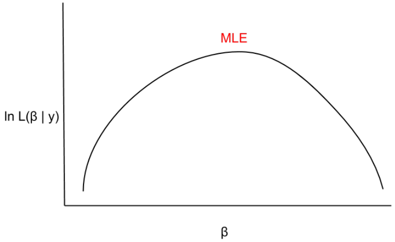

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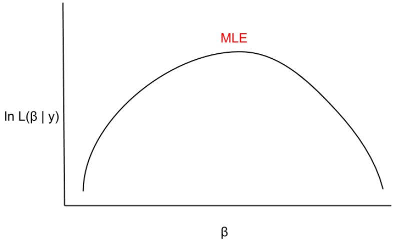

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5. No reason to summarize this curve with only the MLE

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- The curvature at the maximum (standard errors, about which more shortly)

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   - As $n \to \infty$, the distribution of MLE/se(MLE) converges to a Normal.
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Asymptotic Properties of the MLE

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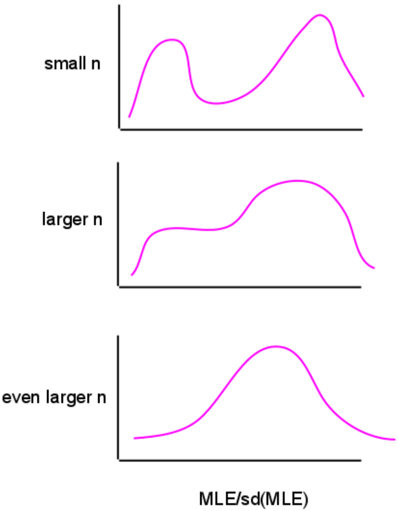
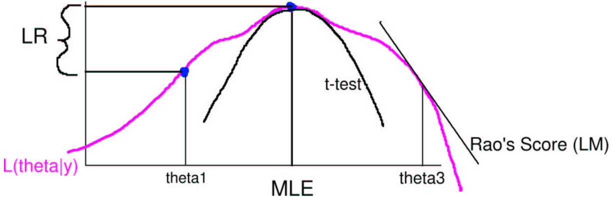
3. Asymptotic efficiency. The MLE contains as much information as can be packed into a point estimator.

Sampling distributions of the MLE: CLT vs LLN



Uncertainty: Likelihood Ratios for nested models



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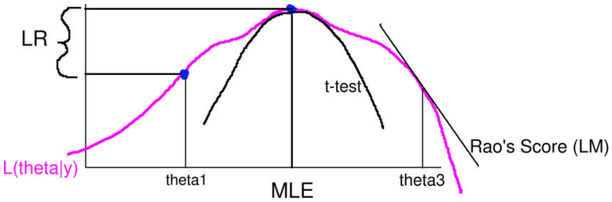

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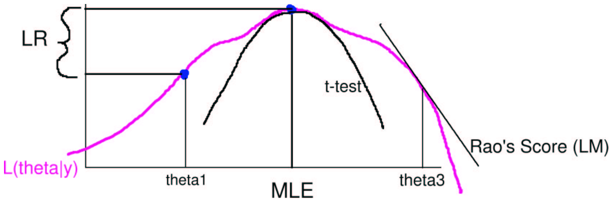

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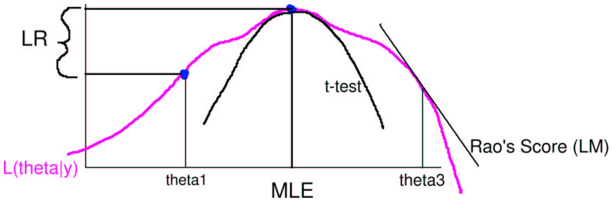

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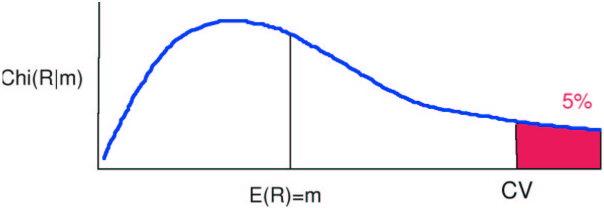

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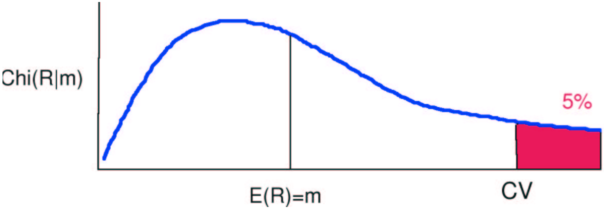

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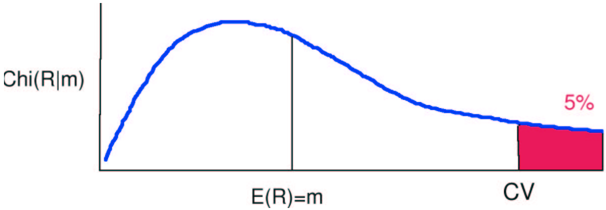

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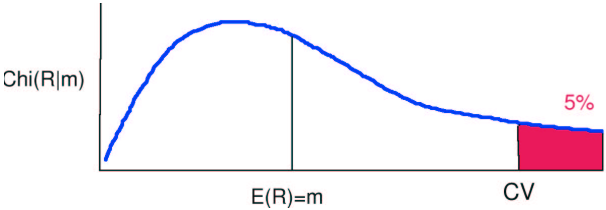

- If restrictions have no effect, $E(R) = m$.
- So only if $r >> m$ will the test parameters be clearly different from zero.
- Disadvantage: Too many likelihood ratio tests may be required to test all points of interest

Standard Errors

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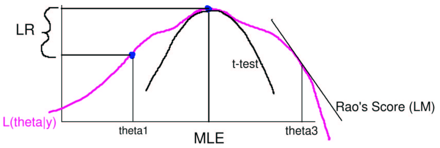

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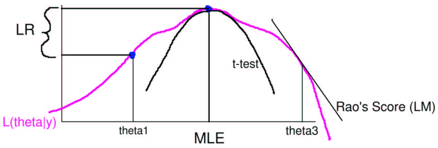

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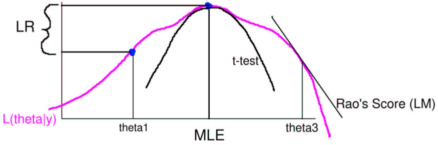

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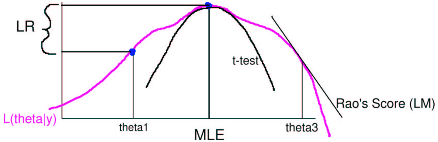

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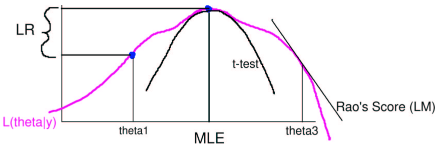

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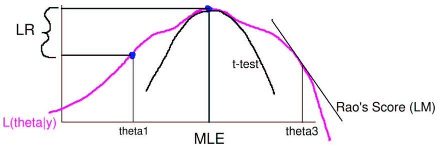

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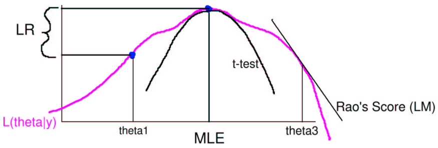

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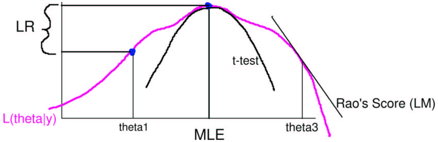

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    - the more information exists in the MLE
    - the larger the likelihood ratio would be in comparing the MLE with *any* other parameter value.

Standard Errors

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E_i	The number of electoral college votes for each state in 2016

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- Mathematical Form:

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- An R function:

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ll.normal <- function(par, X, Y) {
X <- as.matrix(cbind(1, X))
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- Calling it:

```
ll.normal(c(2,1,2,1,33,4,3.2),x,y)
ll.normal(c(2,1,2,1,33,4,3.7),x,y)
ll.normal(c(2,1,2,1,33,4,3.5),x,y)
```

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- Repeat Steps 1–3 $M = 1,000$ times, and plot a histogram of the results.

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Actual Results (calculated before the election) for 1992

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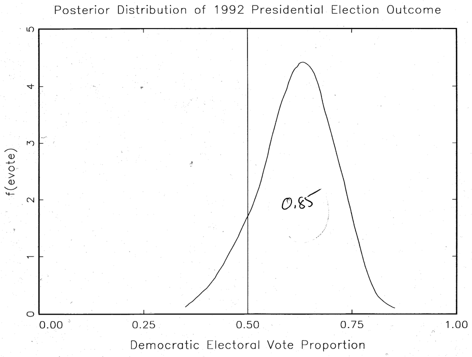

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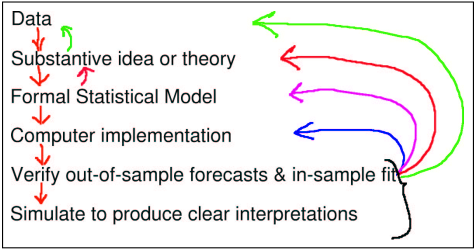
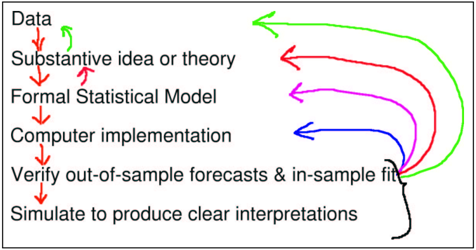
- For what applications would this model be informative?


Data
Substantive idea or theory
Formal Statistical Model
Computer implementation
Verify out-of-sample forecasts & in-sample fit
Simulate to produce clear interpretations

An Outline of the Research Process



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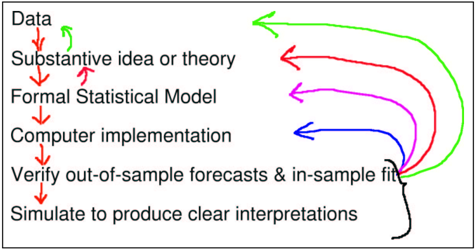
An Outline of the Research Process



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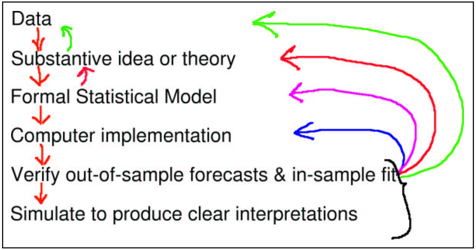

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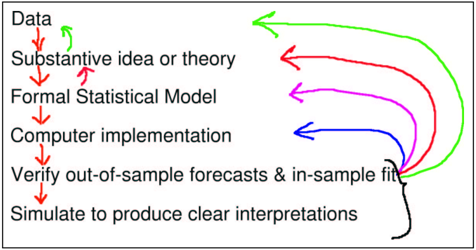

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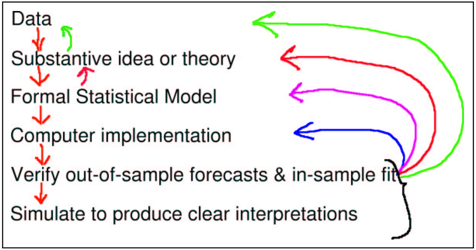

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