Advanced Quantitative Research Methodology, Lecture Notes: Theories of Inference[1]

Gary King
http://GKing.Harvard.Edu

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3. Define: $M = \{M^*, \theta\}$, where M^* is assumed and θ is to be estimated
4. More limited goal: $\mathrm{P}(\theta|y, M^*) \equiv \mathrm{P}(\theta|y)$.

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9. The two differ on the rest

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6. $L(\theta|y)$ is a function: for y fixed at the observed values, it gives the "likelihood" of any value of θ.

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10. The likelihood principle: the data only affect inferences through the likelihood function

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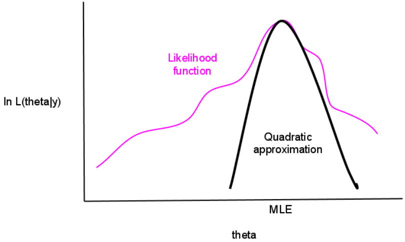

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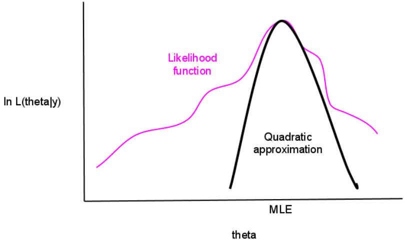

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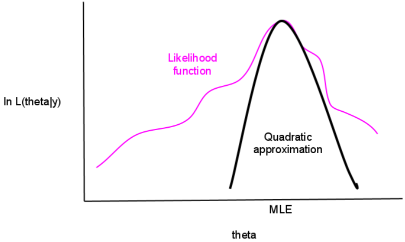

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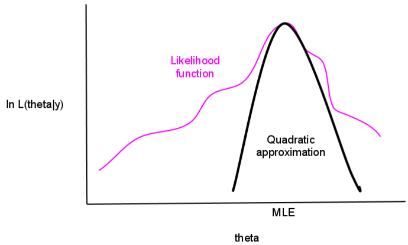

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- Uncertainty of point estimate: curvature at the maximum

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- $P(\theta)$, the prior density — the way Bayes differs from likelihood

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4. A philosophical assumption that nonsample information should matter (as it always does) *and* be formalized and included in all inferences.

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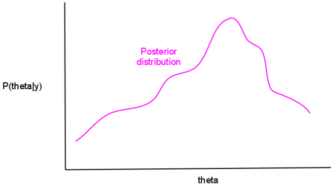
The posterior density, $P(\theta|y)$



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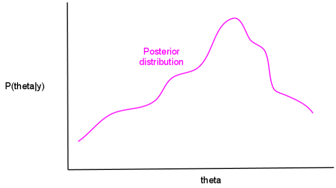

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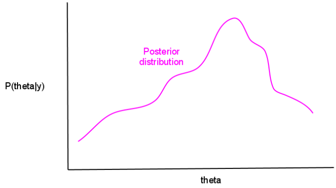

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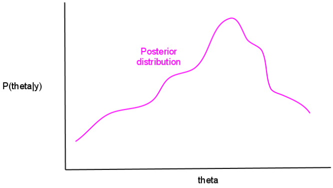

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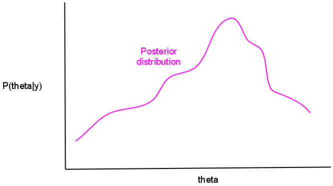

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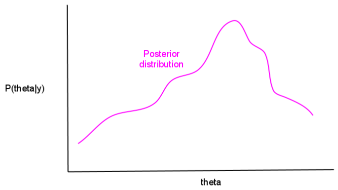

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- If $P(\theta) = 1$, i.e., is uniform in the relevant region, then $L(\theta|y) = P(\theta|y)$.

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- Few fights now between Bayesians and likelihoodists

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   (a) H_0: $\beta = 0$ vs. H_1: $\beta > 0$
   (b) Choose Type I error, probability of deciding H_1 is right when H_0 is really true: say $\alpha = 0.05$

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$$(TS)_\beta|(\beta = 0) \equiv \frac{b - \beta}{\hat{\sigma}_b} \equiv \frac{b}{\hat{\sigma}_b} \sim N(0, 1).$$

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And then first collect your data. You may not revise your hypothesis or your theory.

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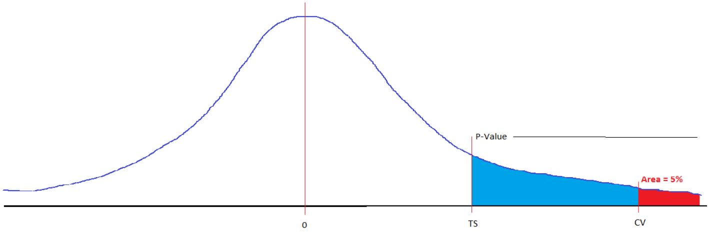

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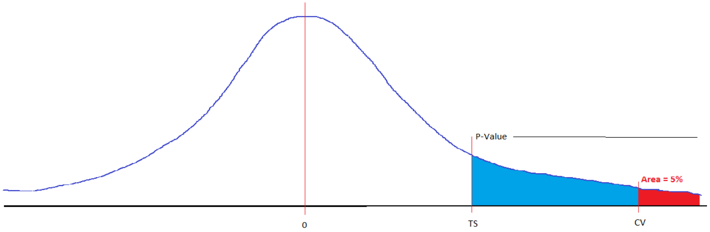

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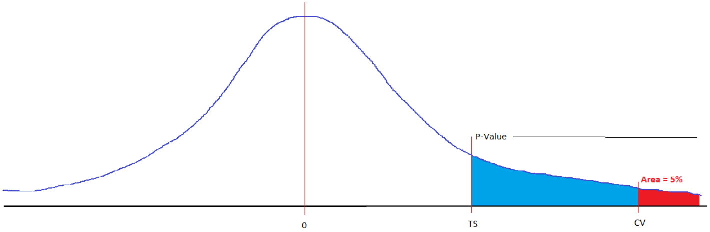

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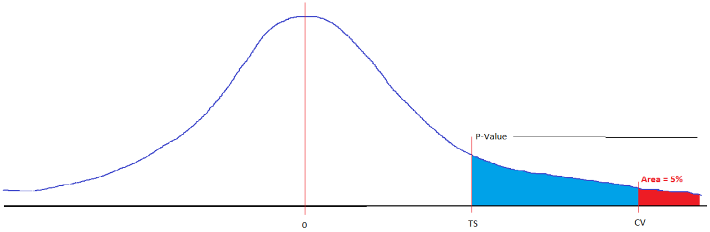

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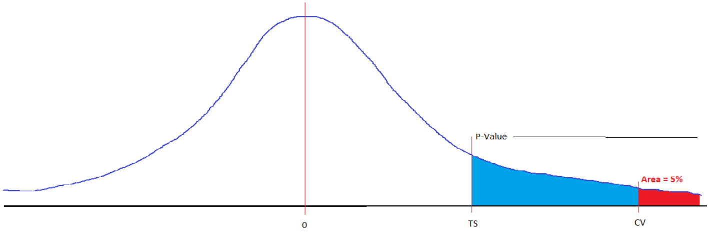

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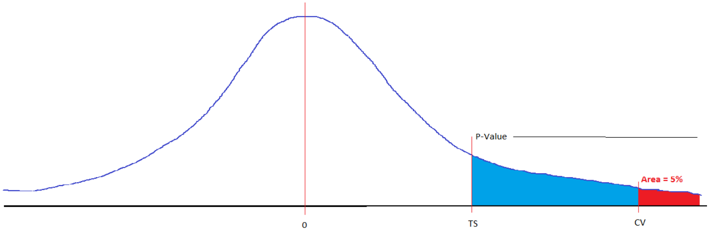

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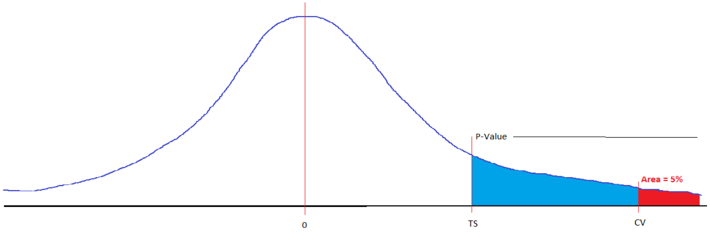

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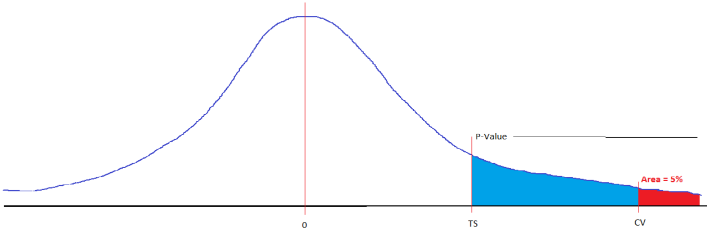

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8. We can use likelihood to compute hypothesis tests and p-values.

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- The key: No assumptions can be trusted; all theories of inference condition on assumptions and so data analysts always struggle trying to understand and get around them

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- What can you do with this density?

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$$L(\beta|y) = k(y) \prod_{i=1}^{n} f_{\text{stn}}(y_i|\beta)$$

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&= \prod_{i=1}^{n}(2\pi)^{-1/2}\exp\left(\frac{-(y_i-\beta)^2}{2}\right)
\end{aligned}
$$

the relative likelihood of β (conditional on the model) having generated the data we observe.

The log-likelihood (Recall: $\ln(ab)=\ln(a)+\ln(b)$):

$$
\begin{aligned}
\ln L(\beta|y) &= \ln[k(y)] + \sum_{i=1}^{n}\ln f_{\text{stn}}(y_i|\beta)\\
&= \ln[k(y)] + \sum_{i=1}^{n}\ln[(2\pi)^{-1/2}] - \sum_{i=1}^{n}\frac{1}{2}(y_i-\beta)^2\\
&\doteq \sum_{i=1}^{n} -\frac{1}{2}(y_i-\beta)^2\\
&= -\frac{1}{2}\sum_{i=1}^{n}(y_i-\beta)^2
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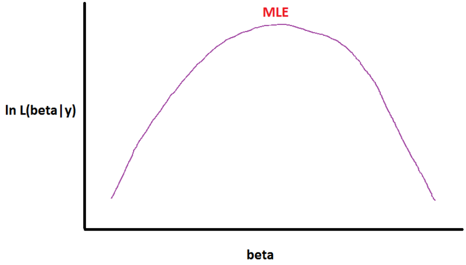
$$

Log-likelihood interpretation



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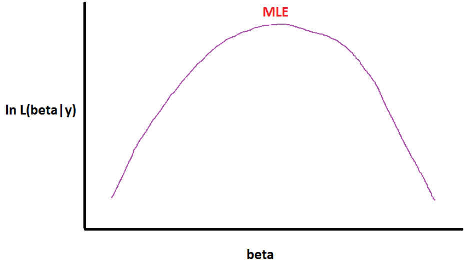

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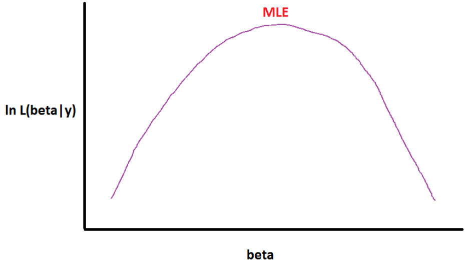

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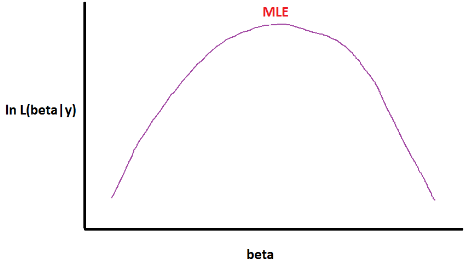

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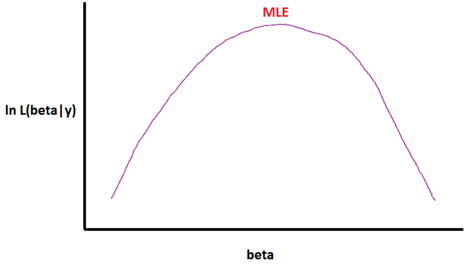

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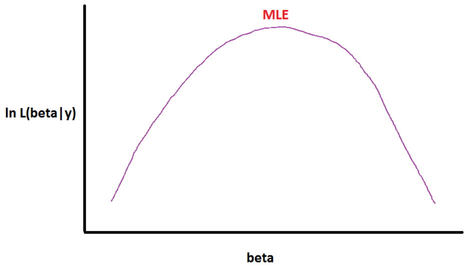

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5. No reason to summarize this curve with only the MLE

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2. Numerically — let the computer do the work for you

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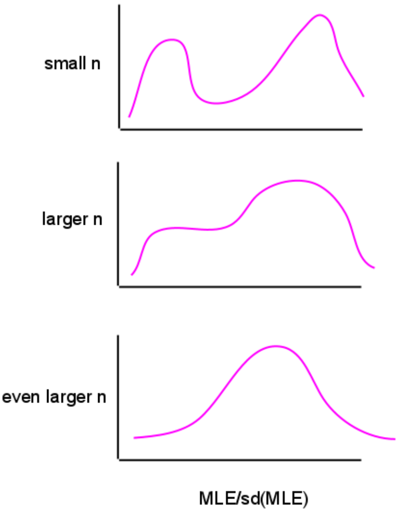
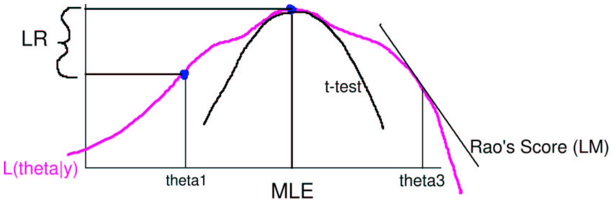
   • Do the LLN and CLT (the 2 most important theorems in statistics) contradict each other?
   (c) Asymptotic efficiency. The MLE contains as much information as can be packed into a point estimator.

Sampling distributions of the MLE: CLT vs LLN



Uncertainty: Likelihood Ratios for nested models



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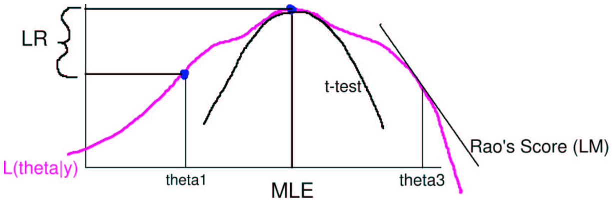

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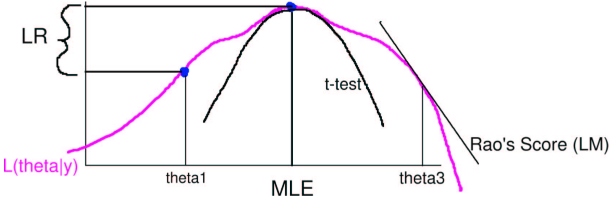

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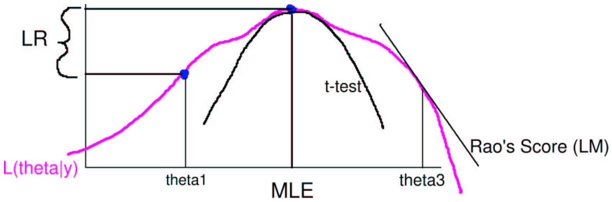

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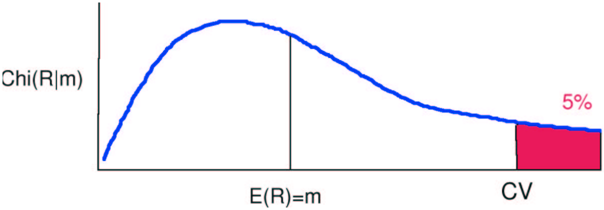

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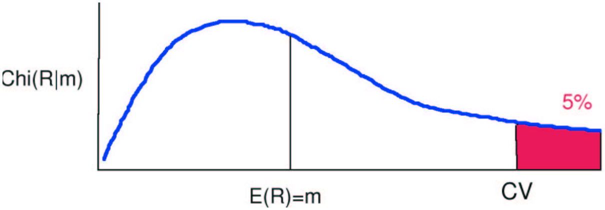

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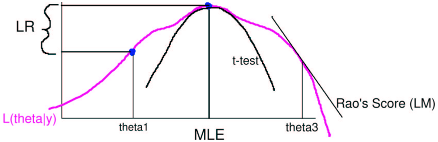

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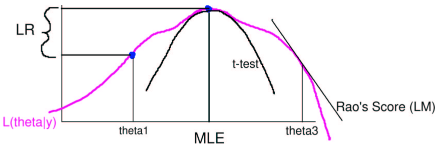

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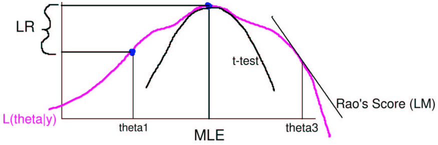

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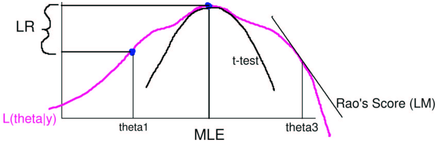

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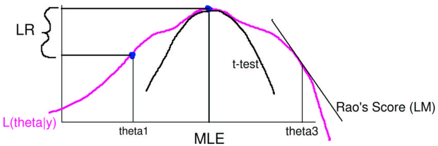

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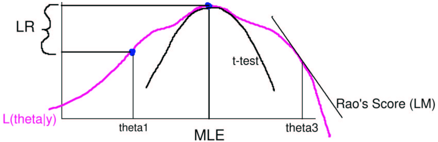

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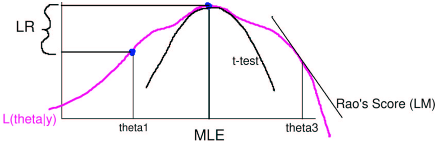

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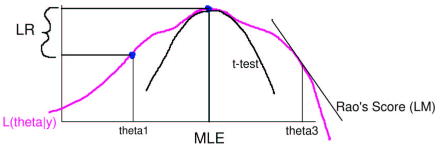

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   (c) The bigger, the larger the likelihood ratio would be in comparing the MLE with *any* other parameter value.

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- We'll discuss later how how to improve the approximation.

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- Calling it:

```
ll.normal(c(2,1,2,1,33,4,3.2),x,y)
ll.normal(c(2,1,2,1,33,4,3.7),x,y)
ll.normal(c(2,1,2,1,33,4,3.5),x,y)
```

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Actual Results (calculated before the election) for 1992

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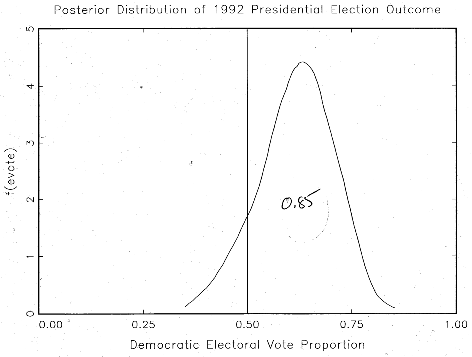

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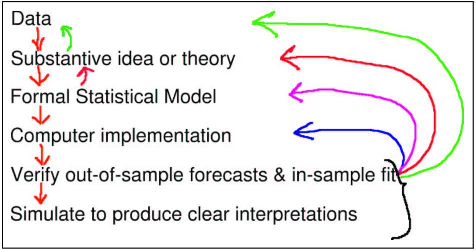
- For what applications would this model be informative?

An Outline of the Research Process



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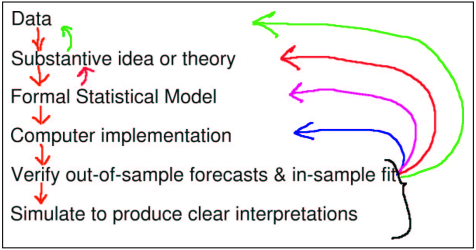

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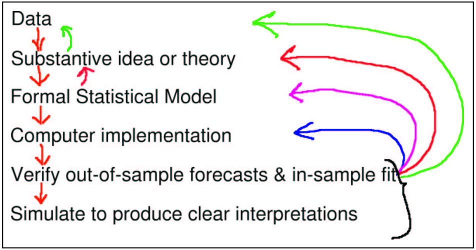

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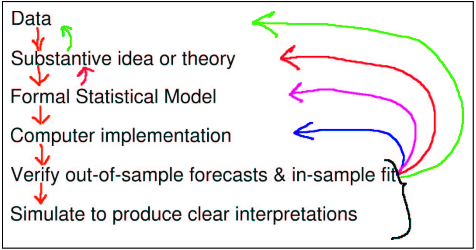

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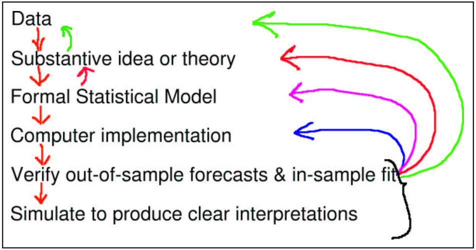

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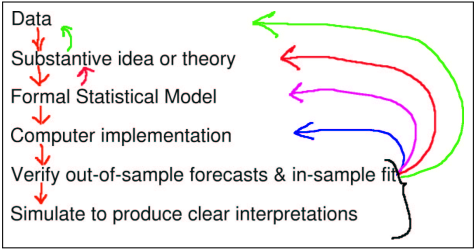

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