

Advanced Quantitative Research Methodology, Lecture Notes: **Introduction**¹

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<http://GKing.Harvard.edu>

February 1, 2015

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- Some of the best experiences here: getting to know people in other fields

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- Do you have the background for this class? A Test: What's this?

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- We cover different amounts of material each week

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④ Focus, like I will, on learning, not grades: Especially when we work on papers, I will treat you like a colleague, not a student

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 - Come whenever you like; if you can't find me or I'm in a meeting, come back, talk to my assistant in the office next to me, or email any time

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- **The number of new methods is increasing fast**
- **Most important methods originate *outside* the discipline of statistics** (random assignment, experimental design, survey research, machine learning, MCMC methods, ...). Statistics: abstracts, proves formal properties, generalizes, and distributes results back out.

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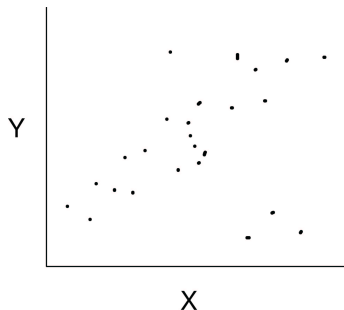
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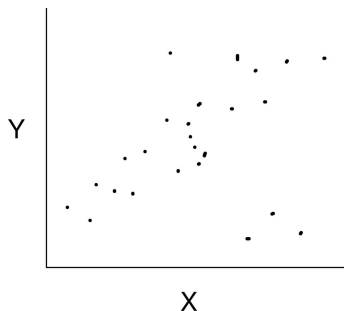
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- This helps us separate the conventions from underlying statistical theory. (How to get an F in Econometrics: follow advice from Psychometrics. Works in reverse too, even when the foundations are identical.)

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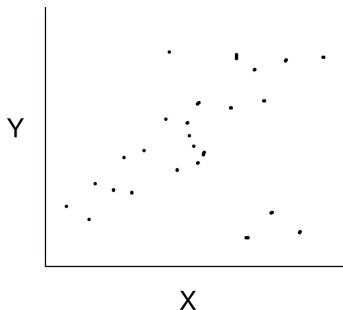


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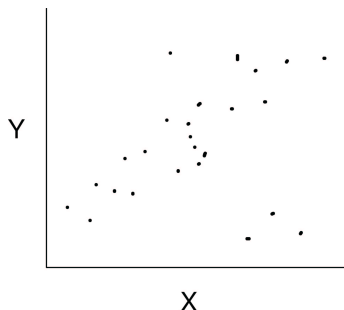
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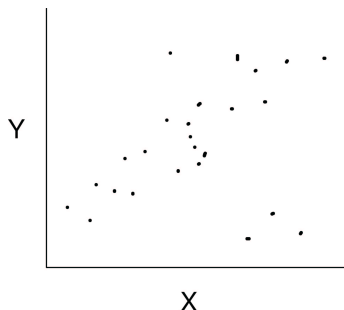
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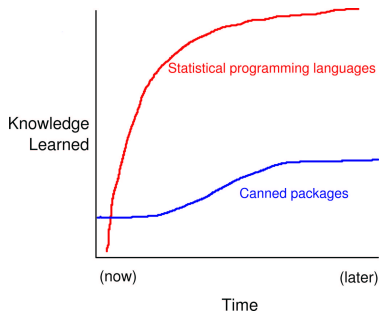
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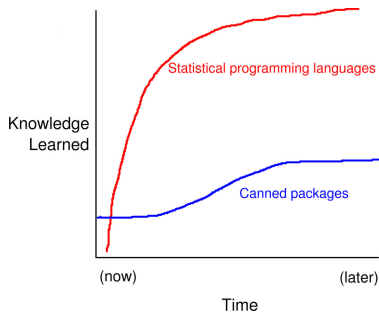
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- from a theory of inference, and for a substantive purpose (like causal estimation, prediction, etc.)

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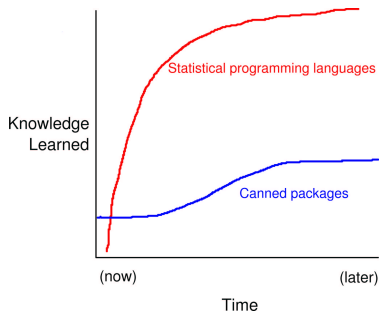


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- We'll use **R** — a free open source program, a commons, a movement
- **Zelig**, which simplifies R and helps you up the steep slope fast (see ZeligProject.org)

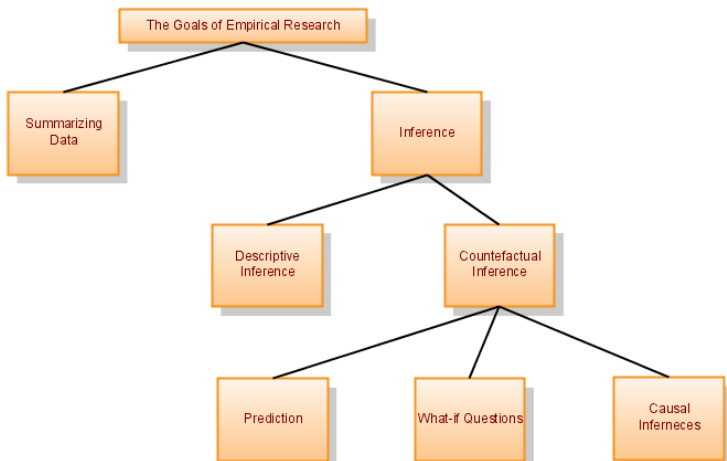
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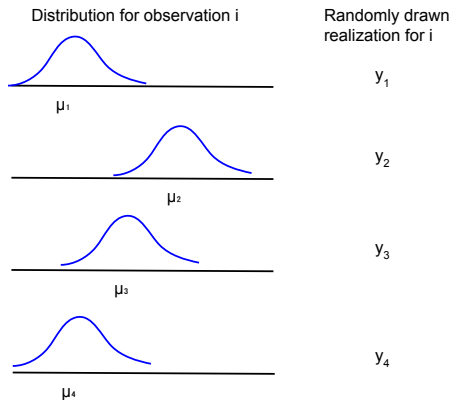
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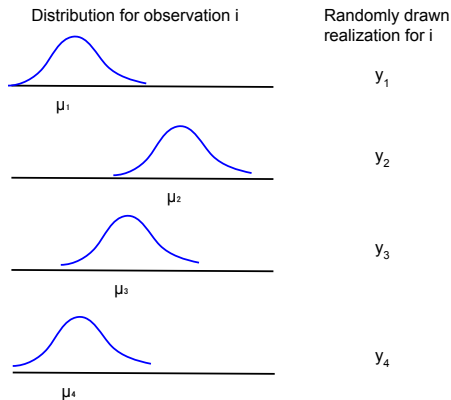
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Is a histogram of y a test of normality?

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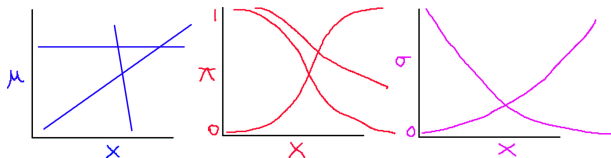
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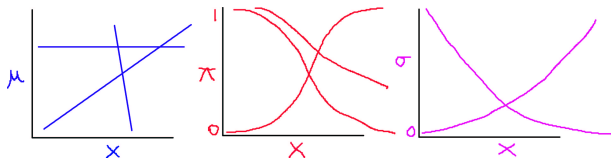
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- (If you know the model, is $R^2 = 1$? Can you predict y perfectly?)

Systematic Components: Examples

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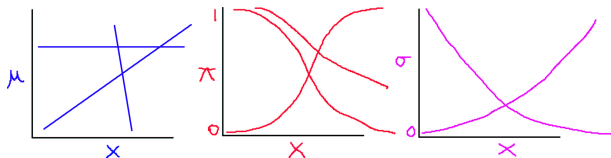


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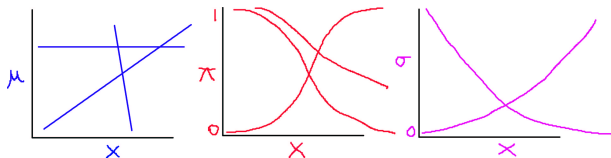
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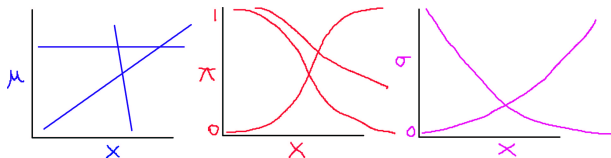
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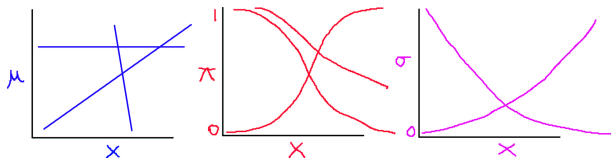
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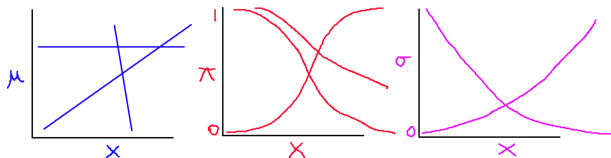
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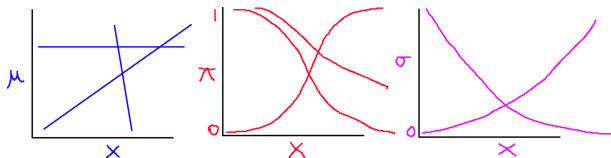
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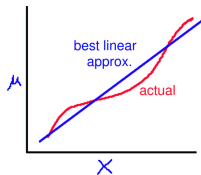
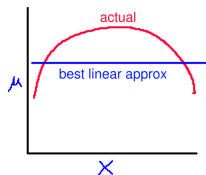
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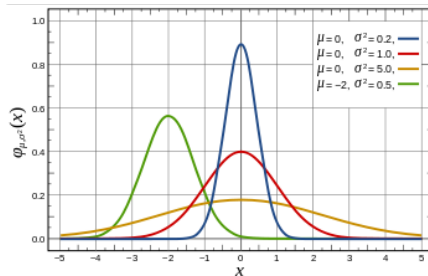
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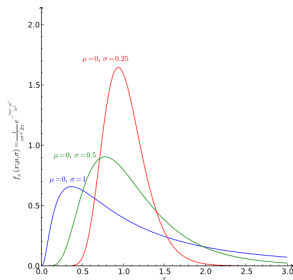
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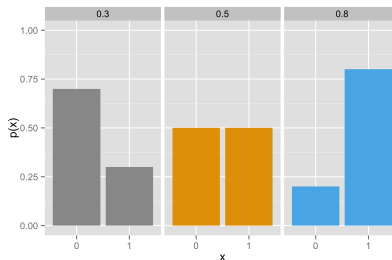
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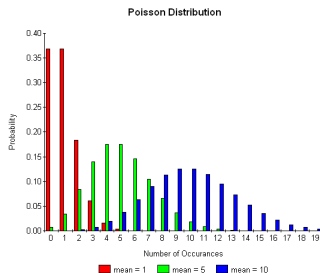
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- Poisson — discrete, countably infinite on the nonnegative integers (for counts)

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 - Ameliorate model dependence: preprocess data (via matching, etc.)

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- Rules can be applied *analytically* or via *simulation*.

Simulation is used to:

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Simulation is used to:

- 1 Solve probability problems
- 2 Evaluate estimators
- 3 Calculate features of probability densities
- 4 Transform statistical results into quantities of interest
- 5 Get the right answer: students get the right answer far more frequently by using simulation than math

What is simulation?

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Survey Sampling

Simulation

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1. Learn about a population by taking a random sample from it

Simulation

1. Learn about a distribution by taking random draws from it

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1. Learn about a population by taking a random sample from it
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4. Example: estimate the mean of the population

Simulation

1. Learn about a distribution by taking random draws from it
2. Use the random draws to approximate a feature of the distribution
3. The approximation is arbitrarily precise for large M
4. Example: Approximate the mean of the distribution

What's the probability that these goats are real?

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HEY, WHO DO YOU THINK YOU'RE LOOKING AT!?



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Monte Hall's Let's Make a Deal

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You have a choice among 3 doors. Behind a random door is a **car**; behind the other two are **goats**. You choose one at random. Monte peeks behind the other two doors and opens the one (or one of the two) with the **goat** and asks if you'd like to switch your door for the other door that hasn't been opened yet. Should you switch?

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doors <- c(1, 2, 3)
```

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sims <- 1000
WinNoSwitch <- 0
WinSwitch <- 0
doors <- c(1, 2, 3)
for (i in 1:sims) {
  WinDoor <- sample(doors, 1)
  choice <- sample(doors, 1)
  if (WinDoor == choice)                                # no switch
    WinNoSwitch <- WinNoSwitch + 1
  doorsLeft <- doors[doors != choice]                   # switch
  if (any(doorLeft == WinDoor))
    WinSwitch <- WinSwitch + 1
}
```

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    WinNoSwitch <- WinNoSwitch + 1
  doorsLeft <- doors[doors != choice]                   # switch
  if (any(doorsLeft == WinDoor))
    WinSwitch <- WinSwitch + 1
}
cat("Prob(Car | no switch)=", WinNoSwitch/sims, "\n")
cat("Prob(Car | switch)=", WinSwitch/sims, "\n")
```


Let's Make a Deal

$\Pr(\text{car} \text{No Switch})$	$\Pr(\text{car} \text{Switch})$
.324	.676
.345	.655
.320	.680
.327	.673

The Birthday Problem

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Given a room with 24 randomly selected people, what is the probability that at least two have the same birthday?

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sims <- 1000  
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```

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  if (length(unique(room)) < people) # same birthday
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    sameday <- sameday+1
}

cat("Probability of >=2 people having the same birthday:", sameday/sims, "\n")
```

The Birthday Problem

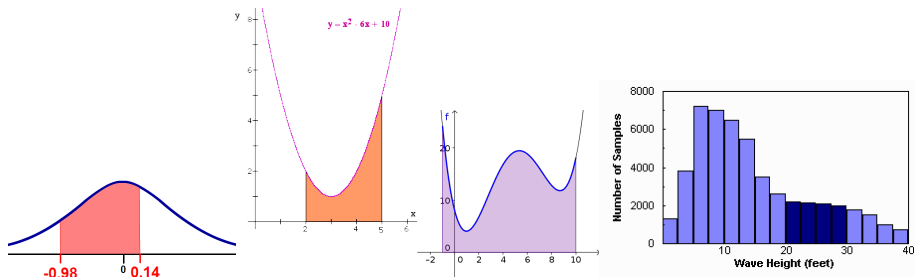
Given a room with 24 randomly selected people, what is the probability that at least two have the same birthday?

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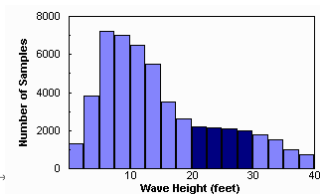
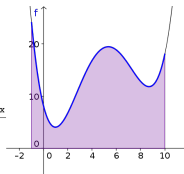
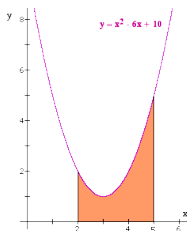
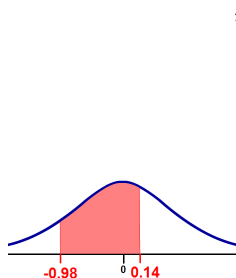
cat("Probability of >=2 people having the same birthday:", sameday/sims, "\n")
```

Four runs: .538, .550, .547, .524

Computing Probabilities from PDFs

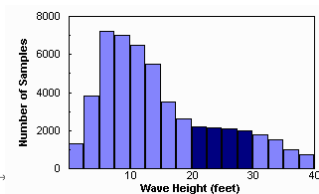
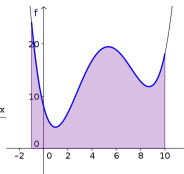
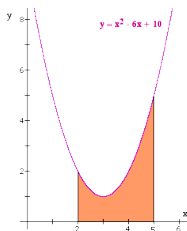
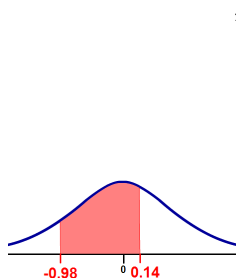


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Required for a PDF:

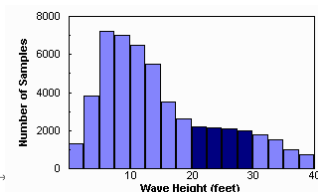
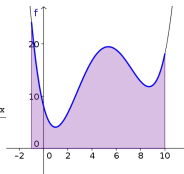
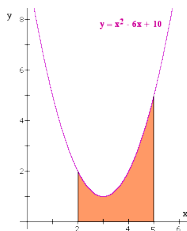
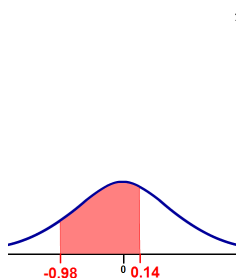
Computing Probabilities from PDFs



Required for a PDF:

$$1 \quad \sum_y P(y) = 1$$

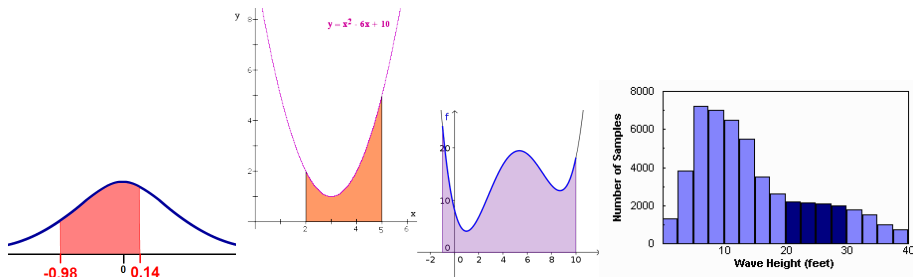
Computing Probabilities from PDFs



Required for a PDF:

① $\sum_y P(y) = 1$ or $\int_{-\infty}^{\infty} P(y) dy = 1$

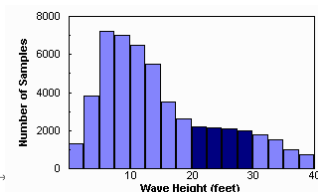
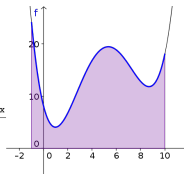
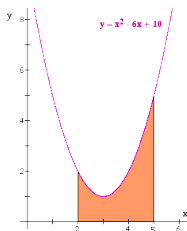
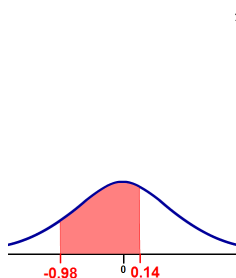
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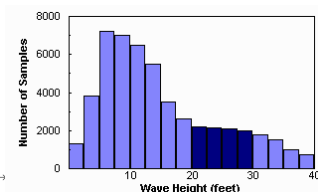
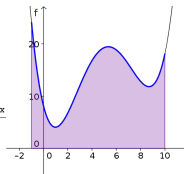
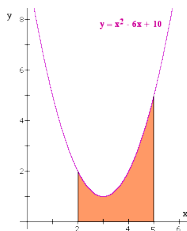
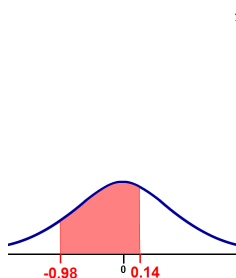


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PDF Calculations:

Computing Probabilities from PDFs



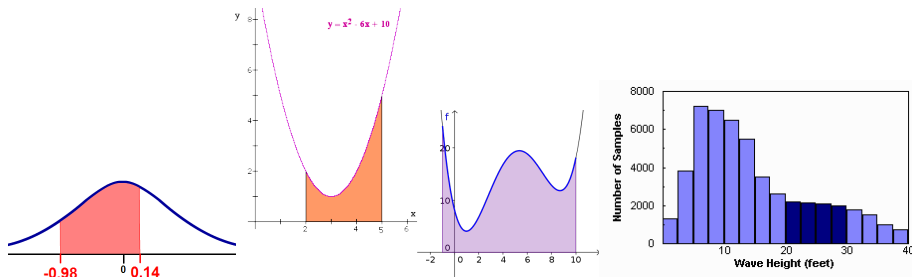
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PDF Calculations:

$$\bullet \Pr(a \leq Y \leq b) = \int_a^b P(y) dy$$

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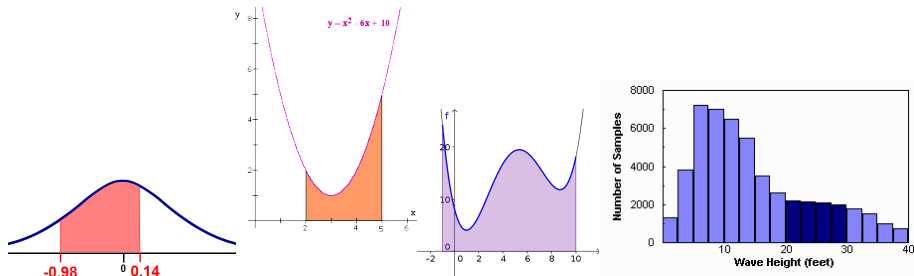
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PDF Calculations:

- $\Pr(a \leq Y \leq b) = \int_a^b P(y) dy$
- For discrete: $\Pr(y) = P(y)$
- For continuous: $\Pr(y) = 0$ (why?)

What you should know about every pdf

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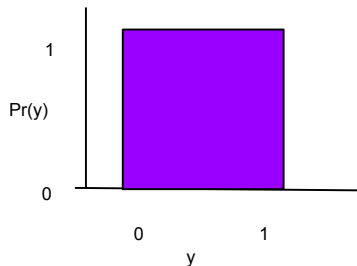
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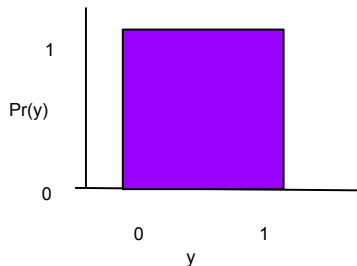
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- How to **verify** that the final expression is indeed a proper density

Uniform Density on the interval $[0, 1]$

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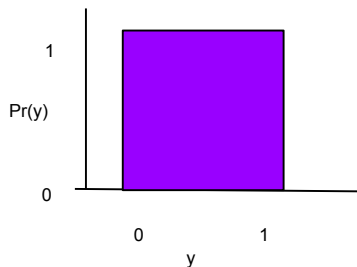


Uniform Density on the interval $[0, 1]$



First Principles about the process that generates Y_i is such that

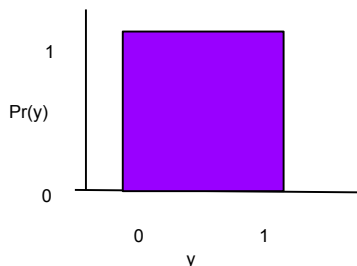
Uniform Density on the interval $[0, 1]$



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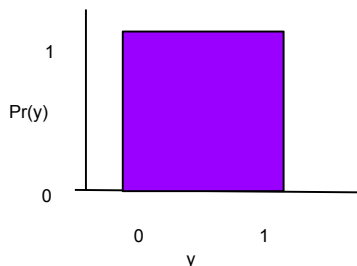
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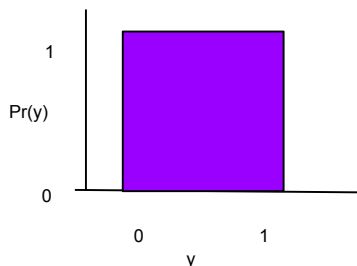
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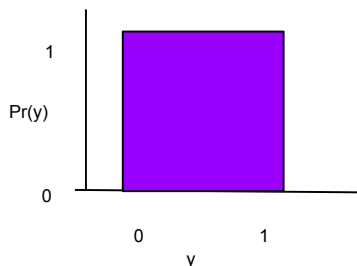
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- How to simulate?

Uniform Density on the interval $[0, 1]$

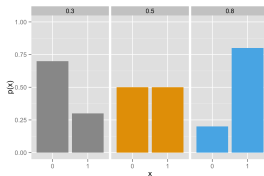


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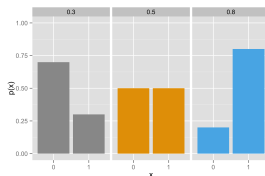
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- How to simulate? `runif(1000)`

Bernoulli pmf

Bernoulli pmf

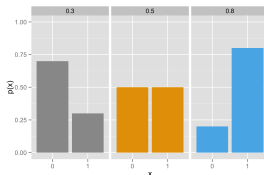


Bernoulli pmf



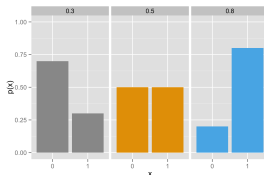
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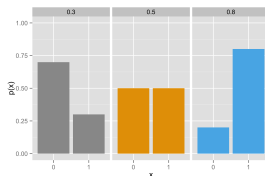
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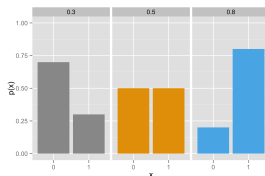
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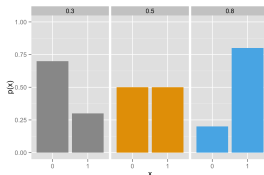
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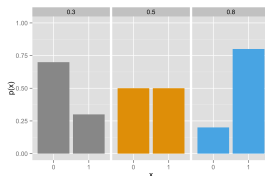
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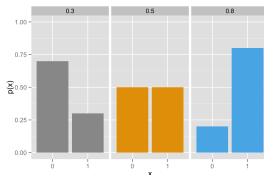
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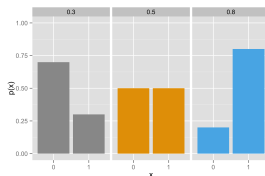
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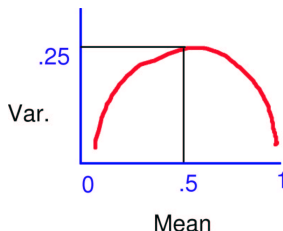
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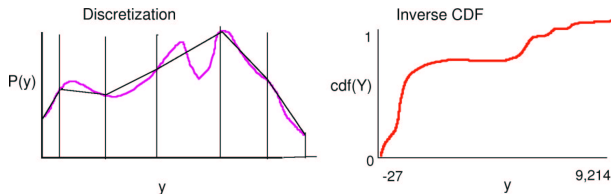
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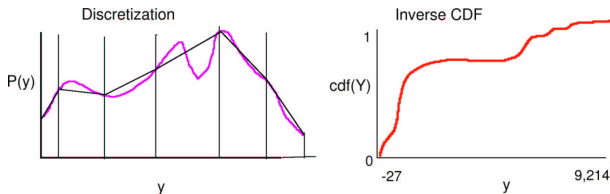
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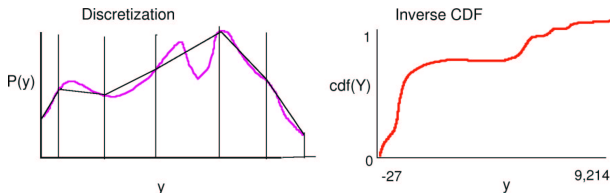


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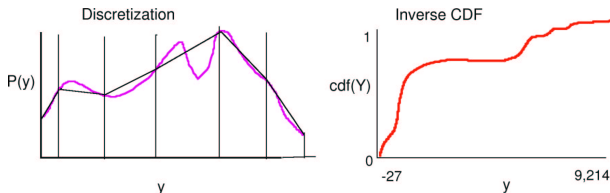
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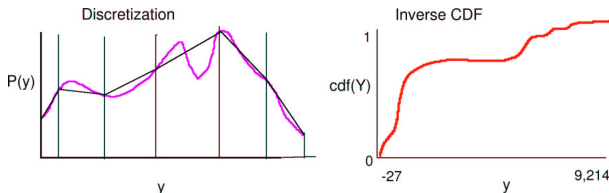
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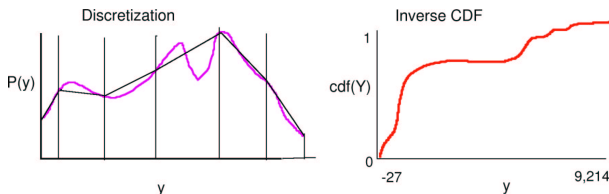
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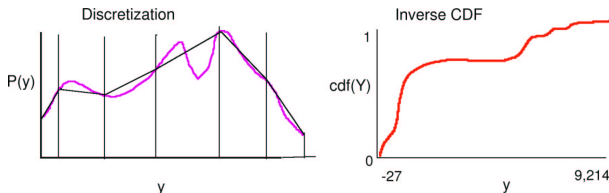
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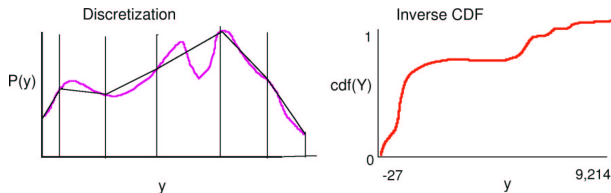
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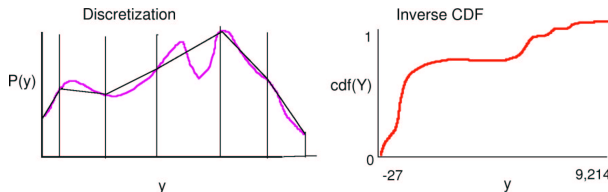


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Inverse CDF: drawing from arbitrary continuous pdfs

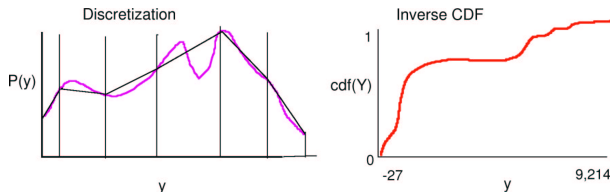


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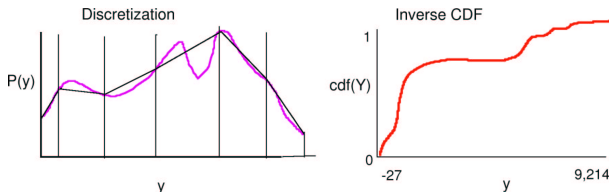
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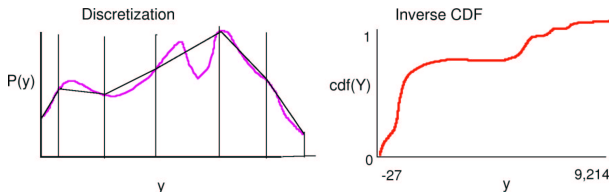
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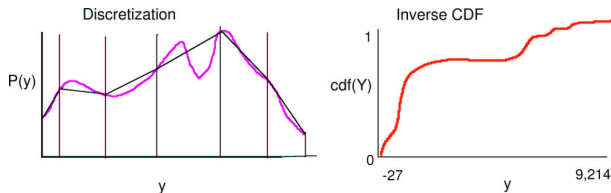
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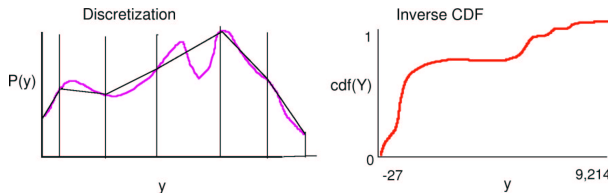


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- Then $F^{-1}(U)$ gives a random draw from $f(Y)$.

Using Inverse CDF to Improve Discretization Method

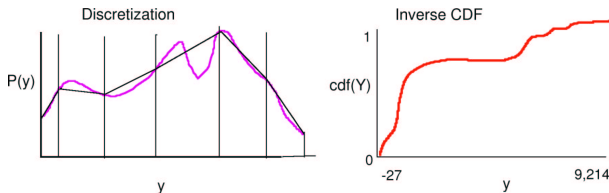


Using Inverse CDF to Improve Discretization Method



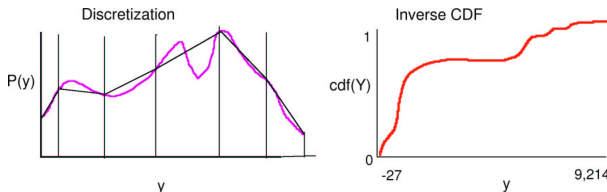
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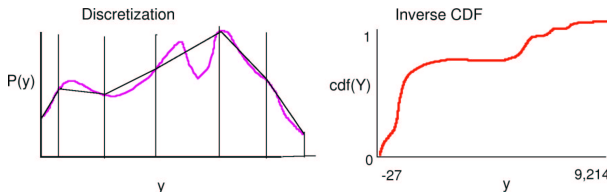
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- **Drawing random numbers from arbitrary multivariate densities: now an enormous literature**

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- Simulating *once* from this density produces k numbers. Special algorithms are used to generate normal random variates (in R, `mvrnorm()`, from the MASS library).

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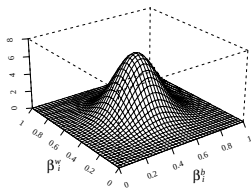
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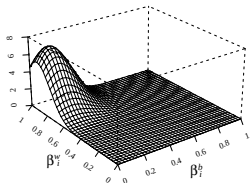
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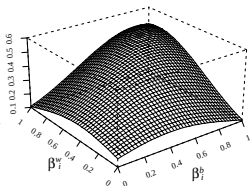
Truncated bivariate normal examples (for β^b and β^w)



(a) 0.5 0.5 0.15 0.15 0



(b) 0.1 0.9 0.15 0.15 0



(c) 0.8 0.8 0.6 0.6 0.5

Parameters are μ_1 , μ_2 , σ_1 , σ_2 , and ρ .

Stop here

We will stop here this year and skip to the next set of slides.
Please refer to the slides below for further information on probability densities and random number generation; they offer more sophisticated .

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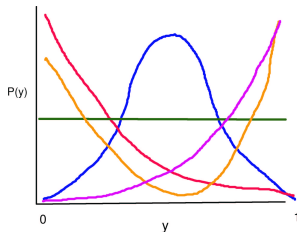
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Reparameterization like this will be key throughout the course.

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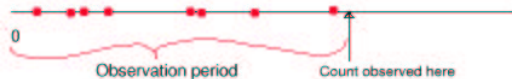
Poisson Distribution

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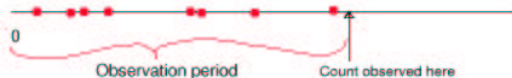
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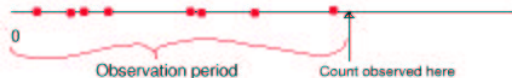
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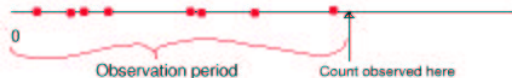
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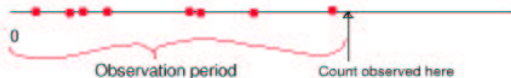
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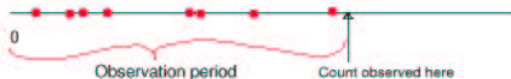
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- No 2 events can occur at the same time

Poisson Distribution

- Begin with an observation period:



- All assumptions are about the events that occur between the start and when we observe the count. The process of event generation is assumed not observed.
- 0 events occur at the start of the period
- Only observe number of events at the end of the period
- No 2 events can occur at the same time
- $\Pr(\text{event at time } t \mid \text{all events up to time } t - 1)$ is constant for all t .

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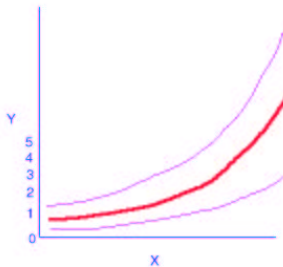
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- How to simulate? We'll use canned random number generators.

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$$\text{gamma}(y|\phi, \sigma^2) = \frac{y^{\phi(\sigma^2-1)^{-1}-1} e^{-y(\sigma^2-1)^{-1}}}{\Gamma[\phi(\sigma^2-1)^{-1}](\sigma^2-1)^{\phi(\sigma^2-1)^{-1}}}$$

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- Choose $E(Y) = \phi$ and σ^2
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- Draw Y from $\text{Poisson}(y|\tilde{\lambda})$, which gives one draw from the negative binomial.

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