

Advanced Quantitative Research Methodology, Lecture Notes: Introduction¹

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January 27, 2014

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- Some of the best experiences here: getting to know people in other fields

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- Do you have the background for this class? A Test: What's this?

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- We cover different amounts of material each week

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④ Focus, like I will, on learning, not grades: Especially when we work on papers, I will treat you like a colleague, not a student

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- **The number of new methods is increasing fast**
- **Most important methods originate *outside* the discipline of statistics** (random assignment, experimental design, survey research, machine learning, MCMC methods, ...). Statistics: abstracts, proves formal properties, generalizes, and distributes results back out.

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- **Second largest APSA section** (Valuable for the job market!)
- Part of a massive **change in the evidence base of the social sciences**: (a) surveys, (b) end of period government stats, and (c) one-off studies of people, places, or events $\rightsquigarrow$ numerous new types and huge quantities of (big) data

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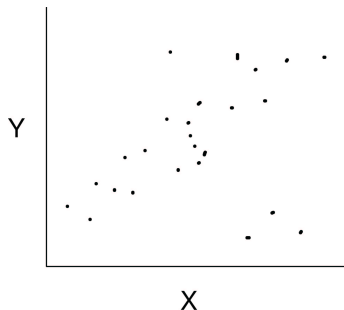
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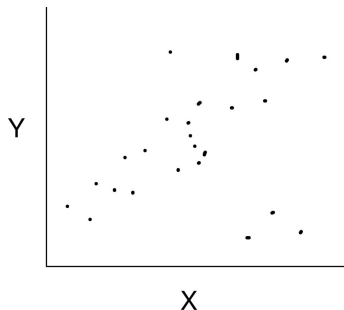
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- This helps us separate the conventions from underlying statistical theory. (How to get an F in Econometrics: follow advice from Psychometrics. Works in reverse too, even when the foundations are identical.)

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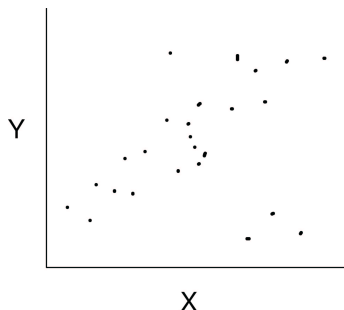


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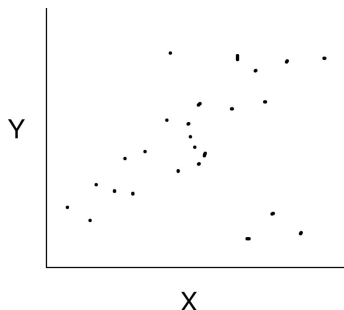
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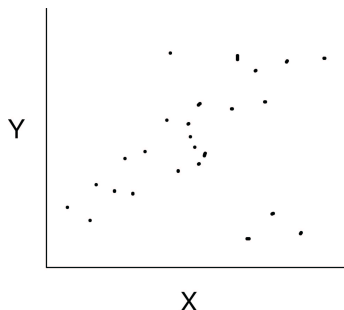
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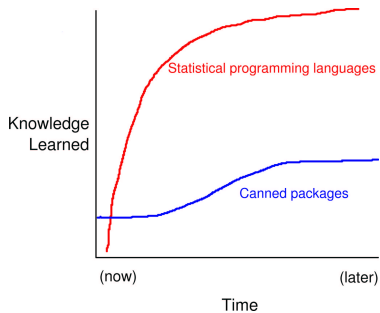
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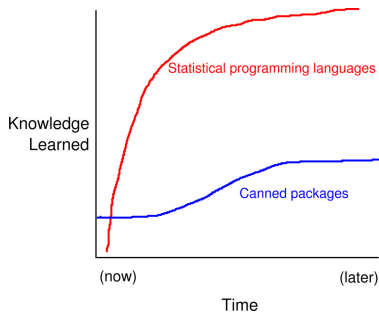
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- **from a theory of inference, and for a substantive purpose** (like causal estimation, prediction, etc.)

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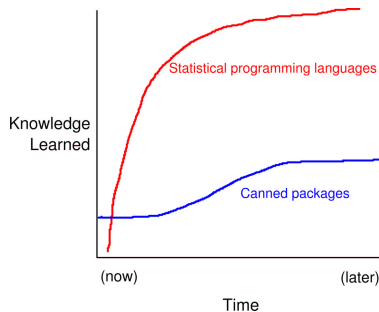


Software options



- We'll use **R** — a free open source program, a commons, a movement

Software options



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- and an R program called **Zelig** (Imai, King, and Lau, 2006-14) which simplifies R and helps you up the steep slope fast (see j.mp/Zelig4)

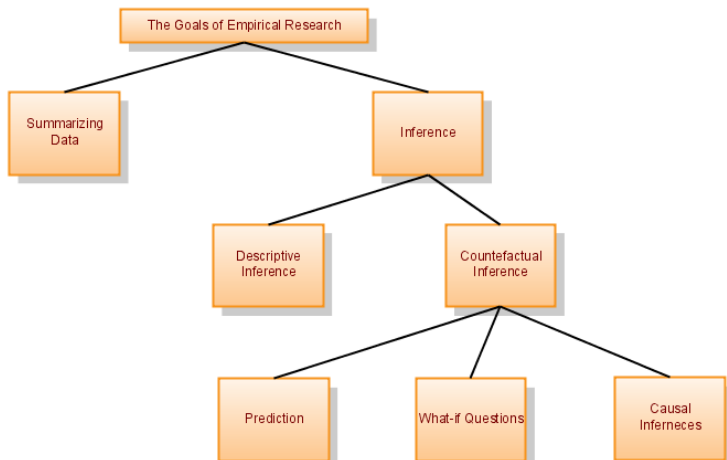
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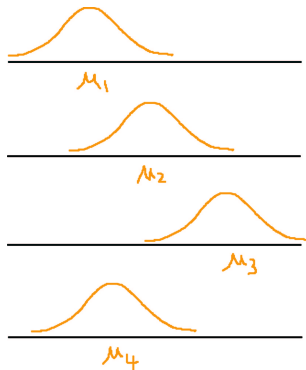
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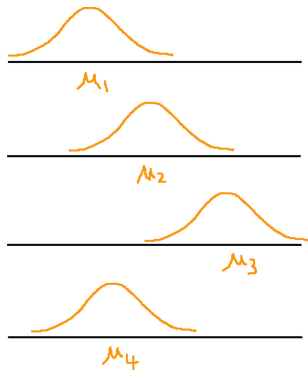
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Is a histogram of y a test of normality?

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- (If you know the model, is $R^2 = 1$? Can you predict y perfectly?)

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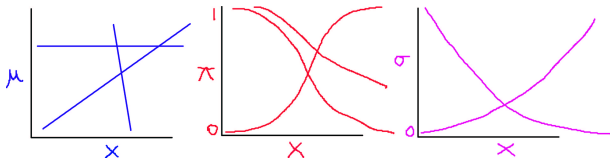
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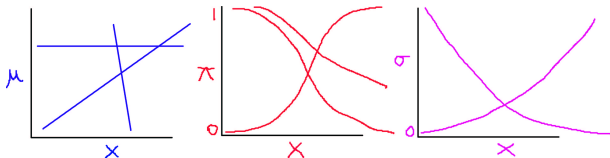
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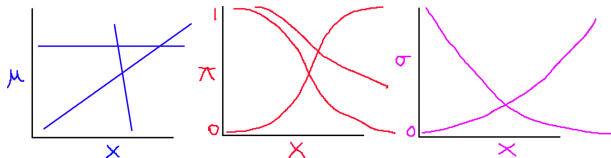


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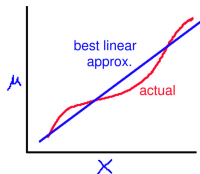
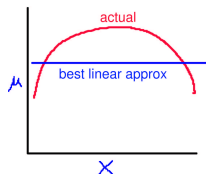
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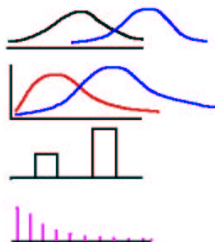
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- Rules can be applied *analytically* or via *simulation*.

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- ⑤ $\rightsquigarrow$ Empirical evidence: students get the right answer far more frequently by using simulation than math

What is simulation?

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Survey Sampling

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1. Learn about a population by taking a random sample from it

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Simulation examples for solving probability problems

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Four runs: .538, .550, .547, .524

Let's Make a Deal

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In Let's Make a Deal, Monte Hall offers what is behind one of three doors. Behind a random door is a car; behind the other two are goats. You choose one door at random. Monte peeks behind the other two doors and opens the one (or one of the two) with the goat. He asks whether you'd like to switch your door with the other door that hasn't been opened yet. Should you switch?

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doors <- c(1, 2, 3)
for (i in 1:sims) {
  WinDoor <- sample(doors, 1)
  choice <- sample(doors, 1)
  if (WinDoor == choice)                # no switch
    WinNoSwitch <- WinNoSwitch + 1
  doorsLeft <- doors[doors != choice]   # switch
  if (any(doesLeft == WinDoor))
    WinSwitch <- WinSwitch + 1
}
```

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}
cat("Prob(Car | no switch)=", WinNoSwitch/sims, "\n")
cat("Prob(Car | switch)=", WinSwitch/sims, "\n")
```

Let's Make a Deal

$\Pr(\text{car} \text{No Switch})$	$\Pr(\text{car} \text{Switch})$
.324	.676
.345	.655
.320	.680
.327	.673

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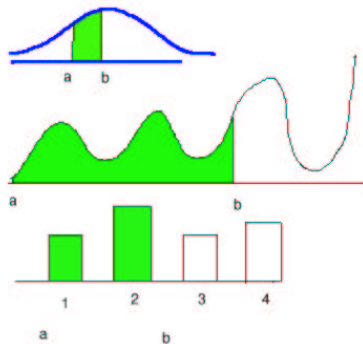
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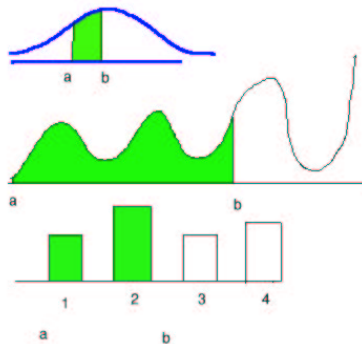
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Computing Probabilities from Densities

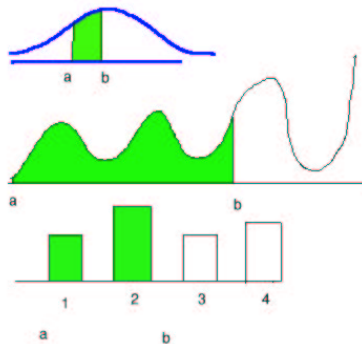


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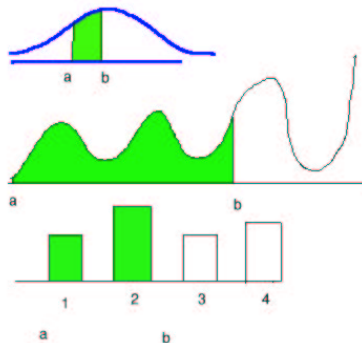
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- For both: $\Pr(a \leq Y \leq b) = \int_a^b P(Y)dY$
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- For continuous: $\Pr(y) = 0$ (why?)

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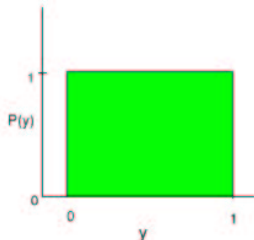
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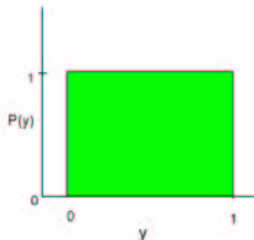
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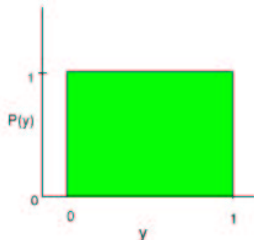


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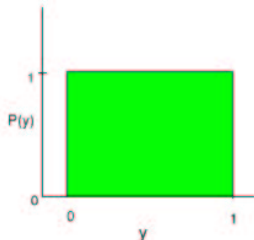


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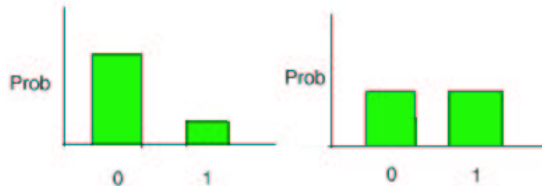
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 - $\implies \Pr(Y_i = y|\pi_i) = \pi_i^y(1 - \pi_i)^{1-y}$
 - Alternative notation: $\Pr(Y_i = y|\pi_i) = \text{Bernoulli}(y|\pi_i) = f_b(y|\pi_i)$

Graphical summary of the Bernoulli



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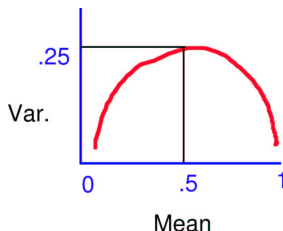
$$\begin{aligned} V(Y) &= E[(Y - E(Y))^2] && \text{(The definition)} \\ &= E(Y^2) - E(Y)^2 && \text{(An easier version)} \\ &= \pi - \pi^2 \\ &= \pi(1 - \pi) \end{aligned}$$

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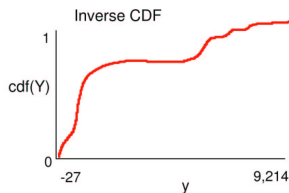
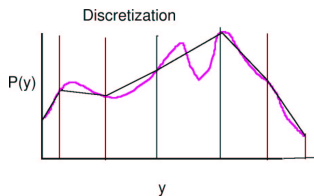
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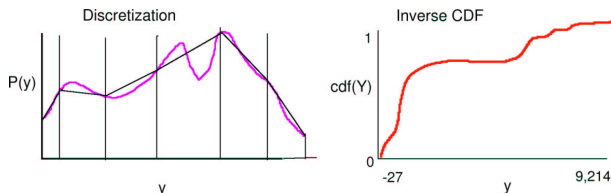
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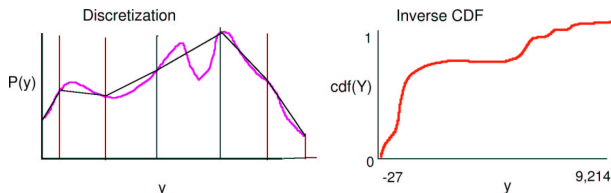


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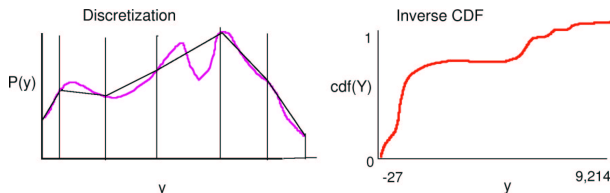
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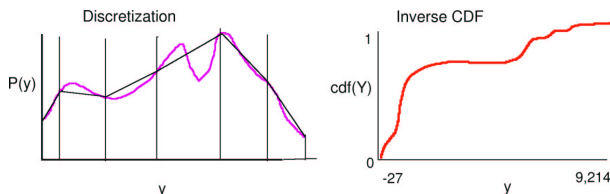
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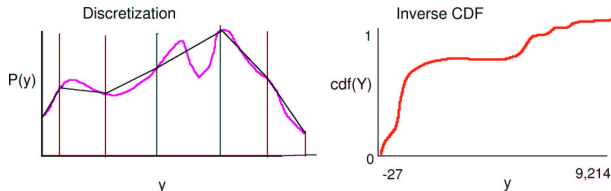
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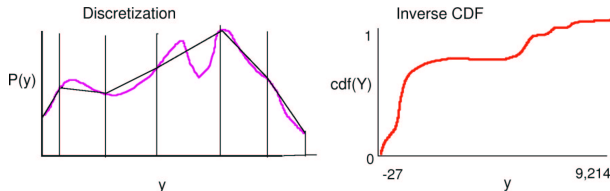


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Inverse CDF method for random draws from continuous pdfs given uniform random numbers

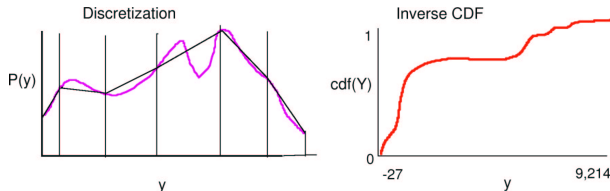


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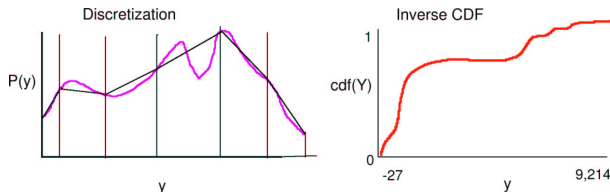
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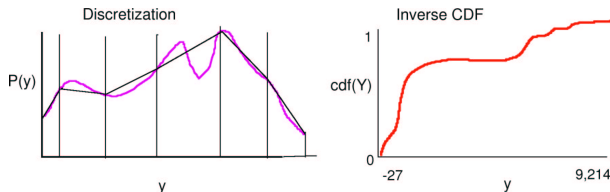
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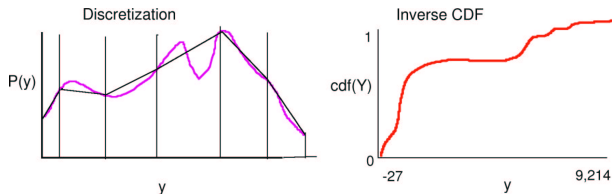
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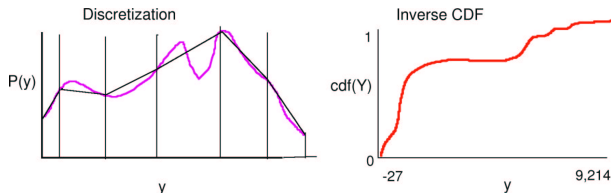


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- Then $F^{-1}(U)$ gives a random draw from $f(Y)$.

A Refined Discretization method

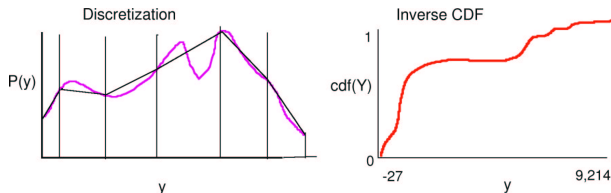


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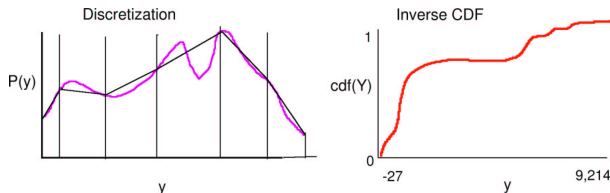
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- Drawing random numbers from arbitrary multivariate densities: now an enormous literature

Stop here

We will stop here this year and skip to the next set of slides. Please refer to the notes below for further information on probability densities and random number generation.

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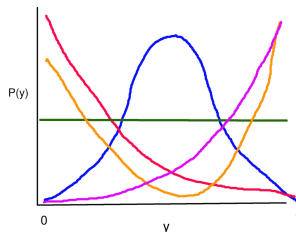
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Reparameterization like this will be key throughout the course.

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- Add up the $\tilde{z}$'s to get $y = \sum_j^N \tilde{z}_j$, which is a draw from the beta-binomial.

Beta-Binomial Analytics

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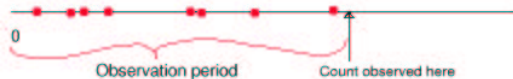
Poisson Distribution

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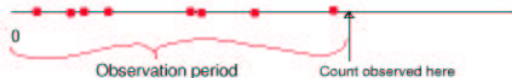
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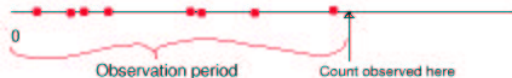
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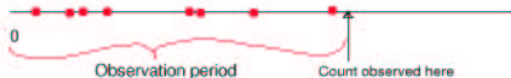
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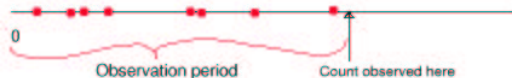
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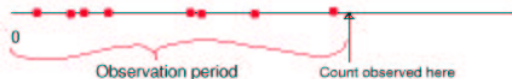
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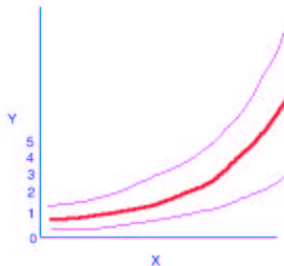
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- Simulating *once* from this density produces k numbers. Special algorithms are used to generate normal random variates (in R, `mvrnorm()`, from the MASS library).

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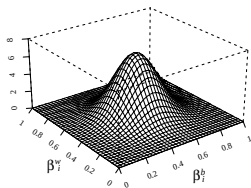
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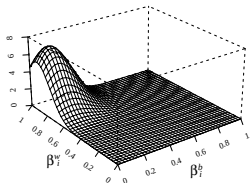
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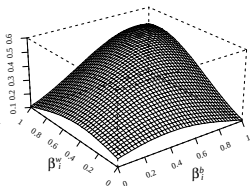
Truncated bivariate normal examples (for β^b and β^w)



(a) 0.5 0.5 0.15 0.15 0



(b) 0.1 0.9 0.15 0.15 0



(c) 0.8 0.8 0.6 0.6 0.5

Parameters are μ_1 , μ_2 , σ_1 , σ_2 , and ρ .

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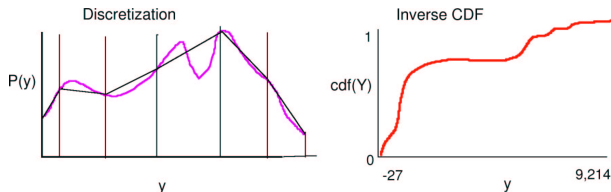
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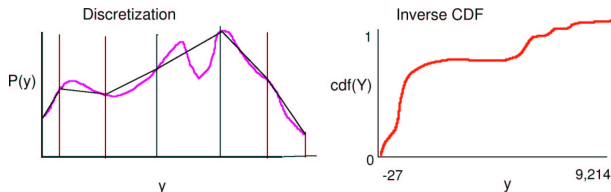
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- Some chips now use quantum effects to create real random numbers.

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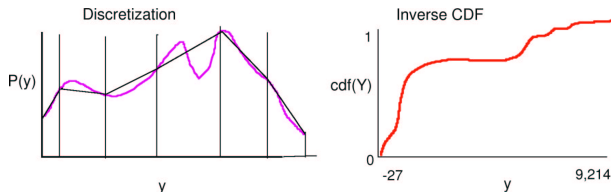


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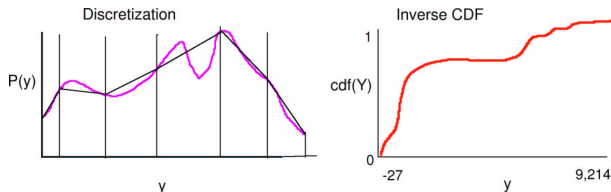
- Divide up PDF into a grid

“Discretization” for random draws from discrete pmfs, given uniform random numbers



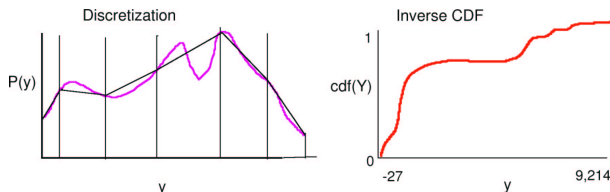
- Divide up PDF into a grid
- Compute by linear approximation area (density/probability) in each interval

“Discretization” for random draws from discrete pmfs, given uniform random numbers



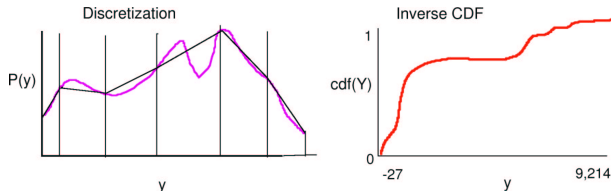
- Divide up PDF into a grid
- Compute by linear approximation area (density/probability) in each interval
- Map $[0,1]$ to the densities proportionally

“Discretization” for random draws from discrete pmfs, given uniform random numbers

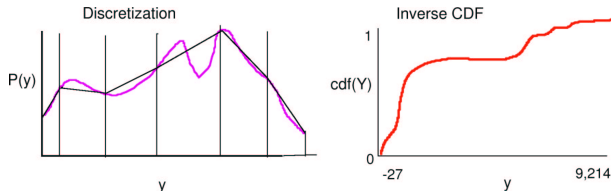


- Divide up PDF into a grid
- Compute by linear approximation area (density/probability) in each interval
- Map $[0,1]$ to the densities proportionally
- Not feasible for multivariate random number generation

Inverse CDF method for random draws from continuous pdfs given uniform random numbers

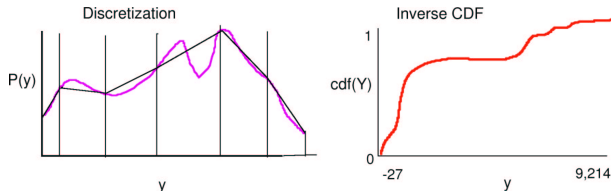


Inverse CDF method for random draws from continuous pdfs given uniform random numbers



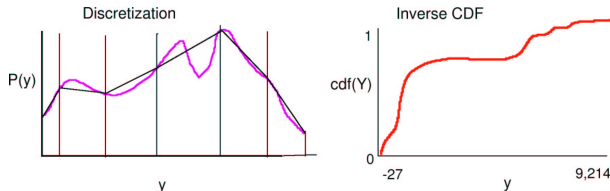
- From the pdf $f(Y)$, compute the cdf:
 $\Pr(Y \leq y) \equiv F(y) = \int_{-\infty}^y f(z) dz$

Inverse CDF method for random draws from continuous pdfs given uniform random numbers



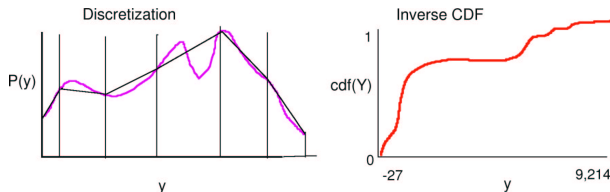
- From the pdf $f(Y)$, compute the cdf:
 $\Pr(Y \leq y) \equiv F(y) = \int_{-\infty}^y f(z) dz$
- Define the inverse cdf $F^{-1}(y)$, such that $F^{-1}[F(y)] = y$

Inverse CDF method for random draws from continuous pdfs given uniform random numbers



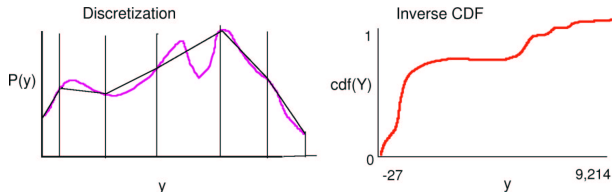
- From the pdf $f(Y)$, compute the cdf:
 $\Pr(Y \leq y) \equiv F(y) = \int_{-\infty}^y f(z) dz$
- Define the inverse cdf $F^{-1}(y)$, such that $F^{-1}[F(y)] = y$
- Draw random uniform number, U

Inverse CDF method for random draws from continuous pdfs given uniform random numbers

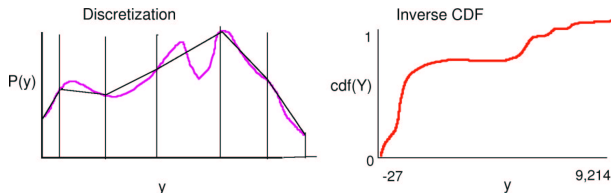


- From the pdf $f(Y)$, compute the cdf:
 $\Pr(Y \leq y) \equiv F(y) = \int_{-\infty}^y f(z) dz$
- Define the inverse cdf $F^{-1}(y)$, such that $F^{-1}[F(y)] = y$
- Draw random uniform number, U
- Then $F^{-1}(U)$ gives a random draw from $f(Y)$.

A Refined Discretization method

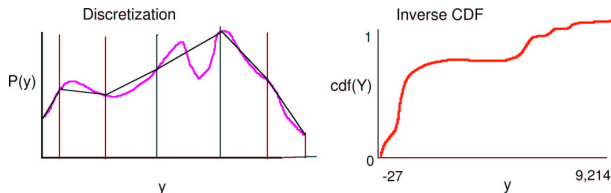


A Refined Discretization method



- Choose interval randomly as above

A Refined Discretization method



- Choose interval randomly as above
- Draw a number within an interval by the inverse CDF method applied to the trapezoidal approximation.