Advanced Quantitative Research Methodology, Lecture Notes: Introduction[1]

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- $\rightsquigarrow$ Some of the best experiences here: getting to know people in other fields

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$$b = (X'X)^{-1}X'y$$

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  - We cover different amounts of material each week

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4. Focus, like I will, on learning (not grades): Especially when we work on papers, I will treat you like a colleague, not a student

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- Most important methods originate *outside* the discipline of statistics (random assignment, experimental design, survey research, machine learning, MCMC methods, . . .). Statistics: abstracts, proves formal properties, generalizes, and distributes results back out.

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- This helps us separate the conventions from underlying statistical theory. (How to get an F in Econometrics: follow advice from Psychometrics. Works in reverse too, even when the foundations are identical.)

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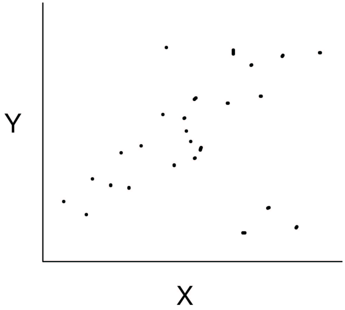

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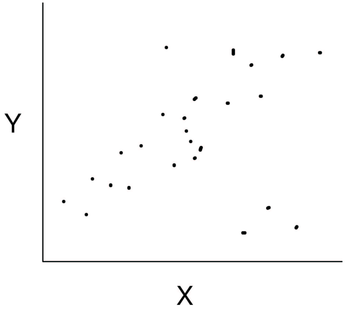

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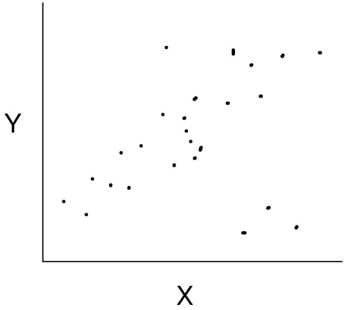

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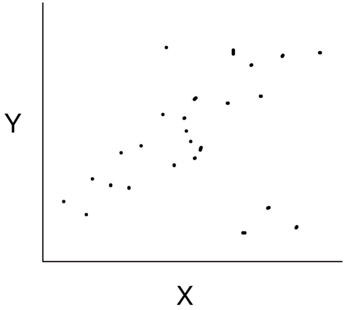

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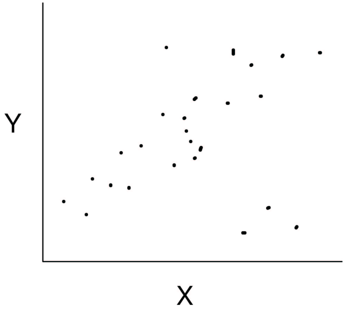

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- from a theory of inference, and for a substantive purpose (like causal estimation, prediction, etc.)

Software options

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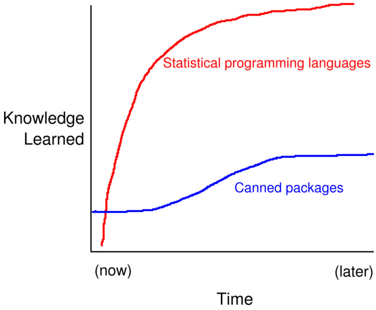

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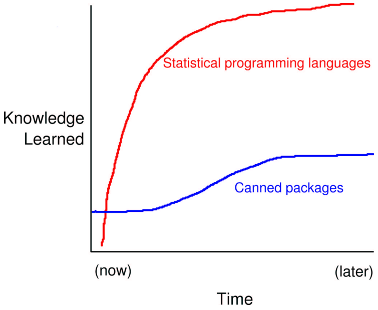

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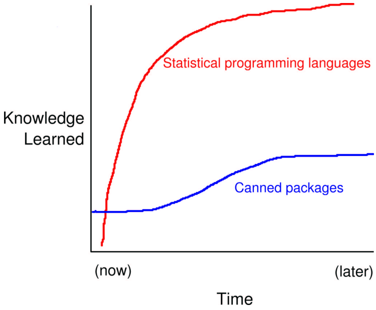

- We'll use R — a free open source program, a commons, a movement
- Zelig, which simplifies R and helps you up the steep slope fast (see ZeligProject.org)

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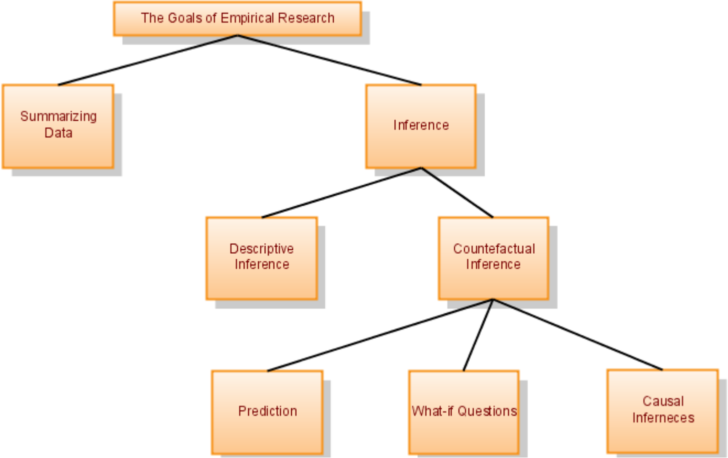

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- Models of dirt on airplanes, vs models of aerodynamics

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  - X is fixed (not random).

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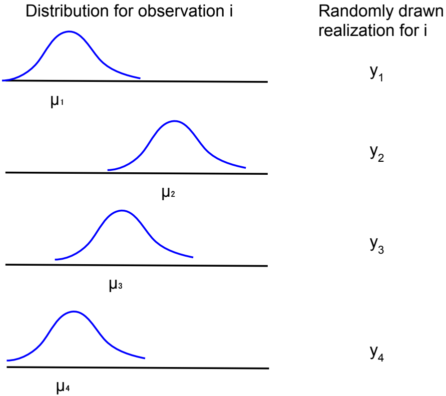

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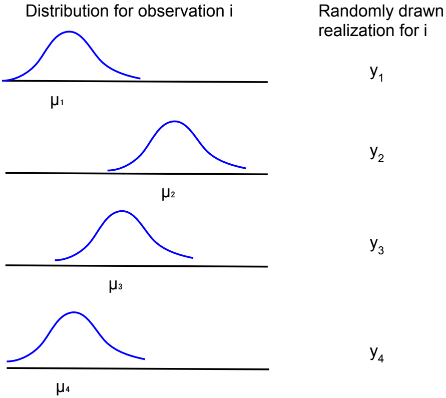

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A Test: Is a histogram of y a test of normality?

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- (A Test: If you know the model, is $R^2 = 1$?)

Systematic Components: Examples

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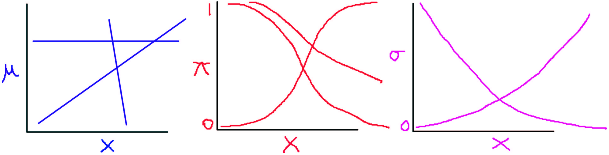

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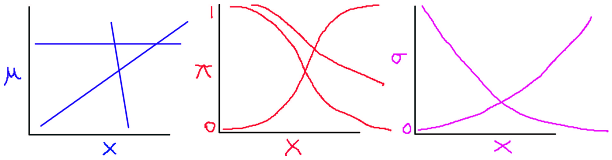

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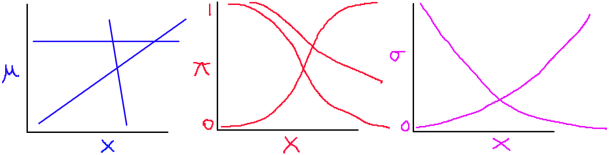

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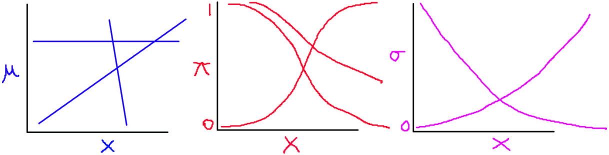

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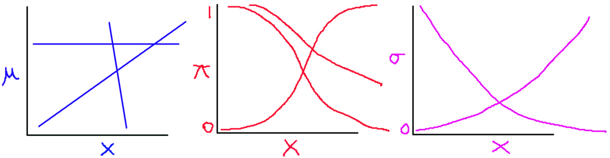

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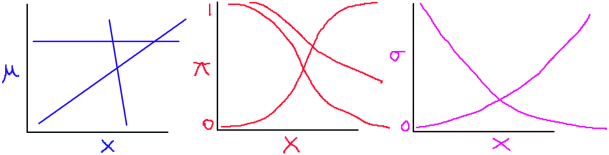

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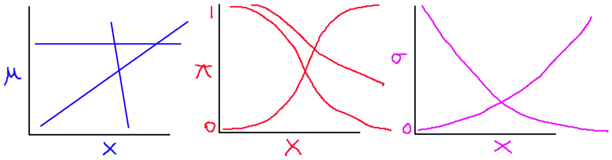

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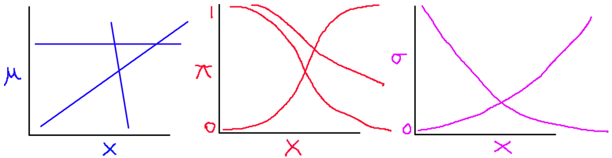

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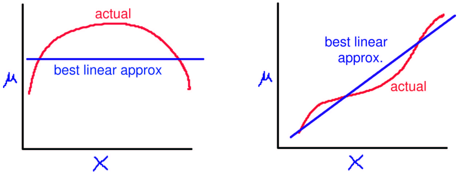

Overview of Stochastic Components

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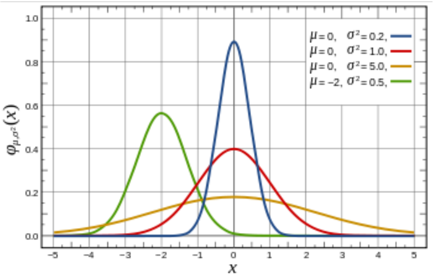

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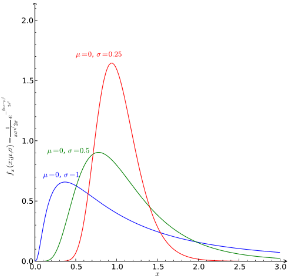

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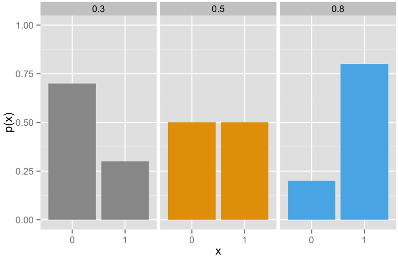

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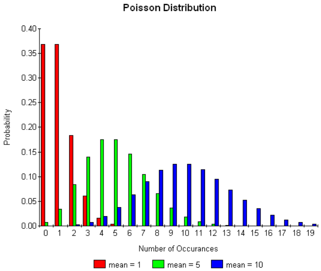

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- Poisson — discrete, countably infinite on the nonnegative integers

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        - Remove model dependence: preprocess data (via matching, etc.)

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- Rules can be applied *analytically* or via *simulation*.

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5. Get the right answer: students get the right answer far more frequently by using simulation than math

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3. The estimate is arbitrarily precise for large n	3. The approximation is arbitrarily precise for large M
4. Example: estimate the mean of the population	4. Example: Approximate the mean of the distribution

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  - $\rightsquigarrow$ Make others justify their choice of a DGP!

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You have a choice among 3 doors. Behind a random door is a car; behind the other two are goats. You choose one at random. Monte peeks behind the other two doors and opens the one (or one of the two) with the goat and asks if you'd like to switch your door for the other door that hasn't been opened yet. Should you switch?

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doors <- c(1, 2, 3)
for (i in 1:sims) {
  WinDoor <- sample(doors, 1)
  choice <- sample(doors, 1)
  if (WinDoor == choice)                          # no switch
     WinNoSwitch <- WinNoSwitch + 1
  doorsLeft <- doors[doors != choice]             # switch
  if (any(doorsLeft == WinDoor))
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  }
cat("Prob(Car | no switch)=", WinNoSwitch/sims, "\n")
cat("Prob(Car | switch)=", WinSwitch/sims, "\n")
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Let's Make a Deal

$\Pr(\text{car}\|\text{No Switch})$	$\Pr(\text{car}\|\text{Switch})$
.324	.676
.345	.655
.320	.680
.327	.673

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```
Four runs: .538, .550, .547, .524
```

Computing Probabilities from PDFs

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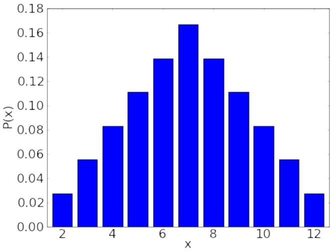

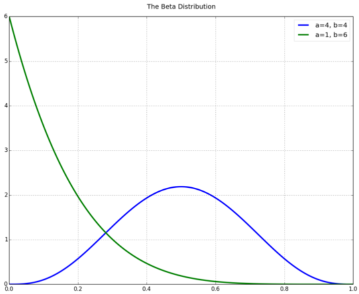

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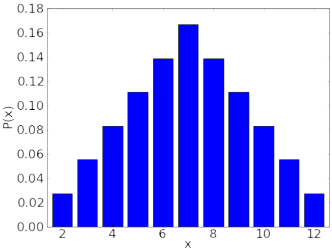

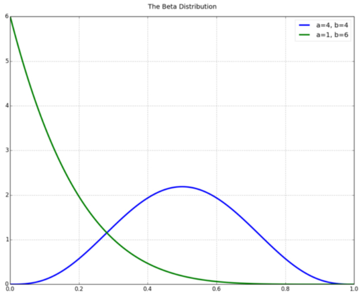

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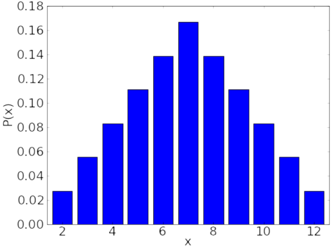

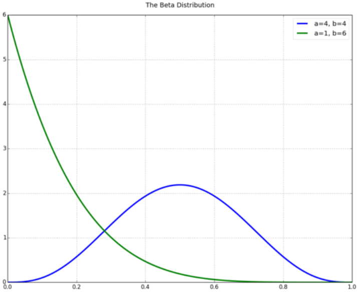

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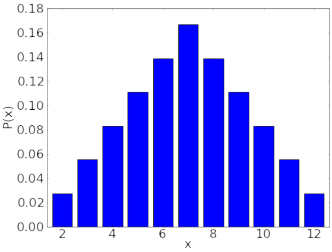

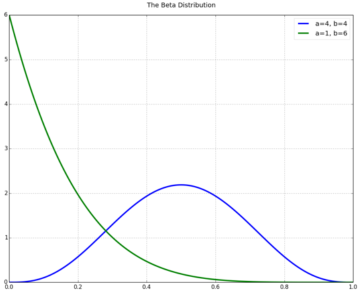

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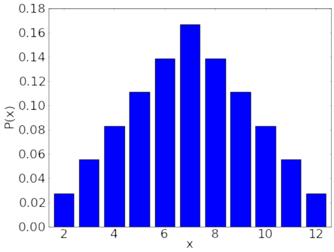

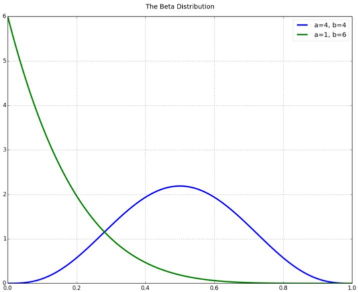

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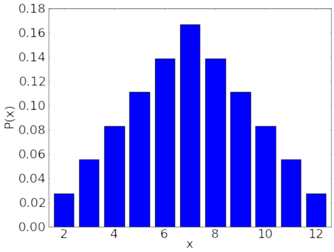

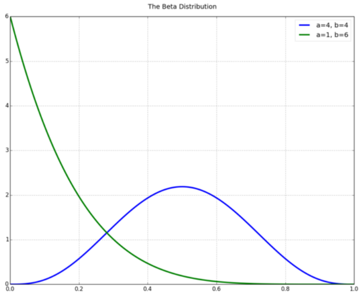

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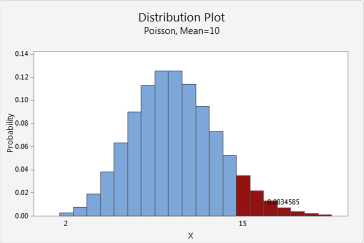

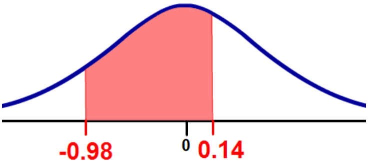

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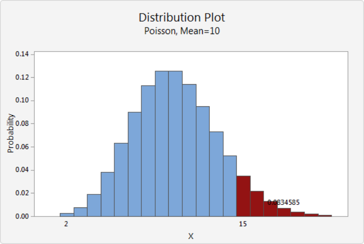

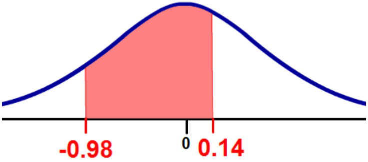

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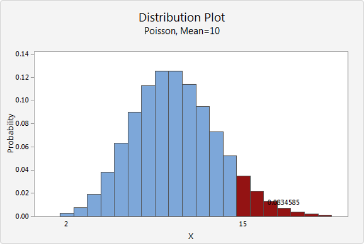

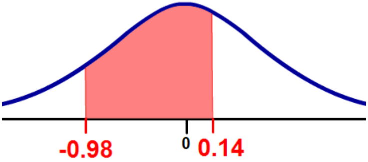

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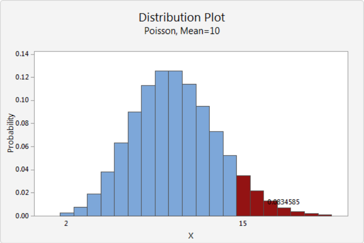

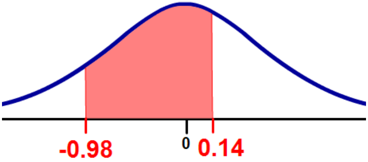

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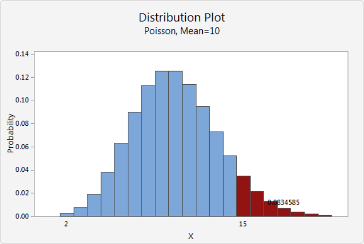

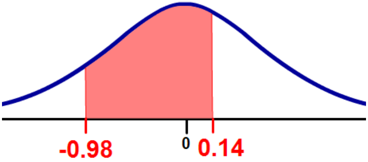

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What you should know about every pdf

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Uniform Density on the interval $[0, 1]$

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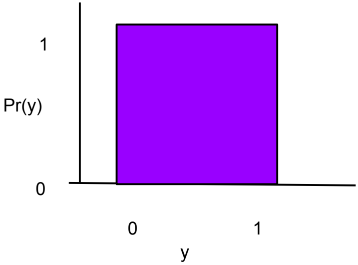

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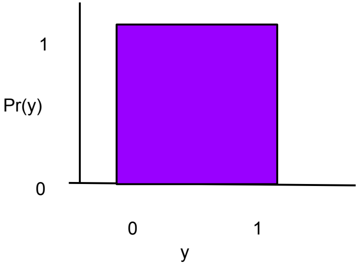

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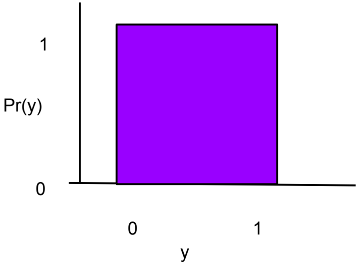

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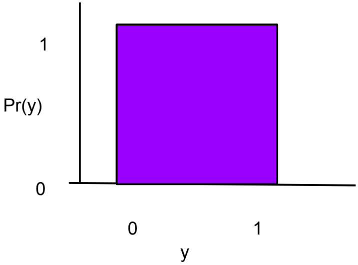

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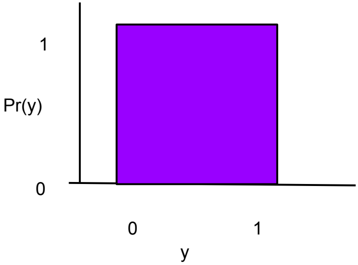

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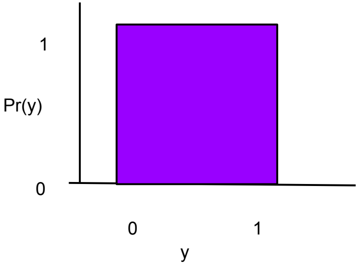

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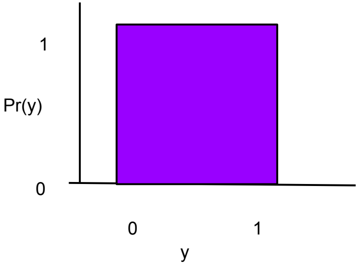

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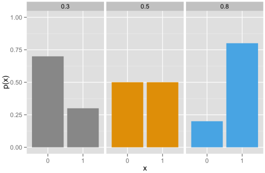

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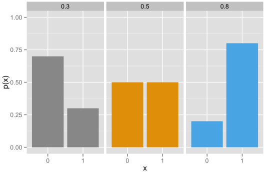

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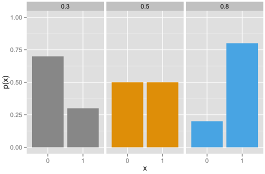

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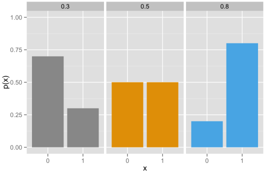

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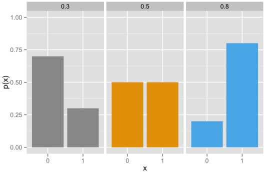

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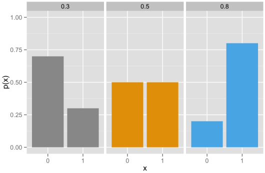

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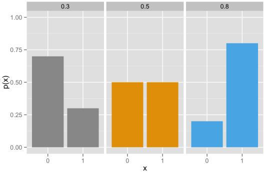

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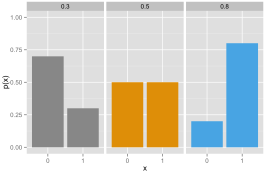

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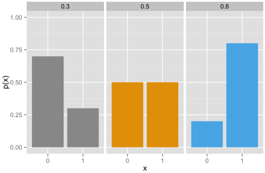

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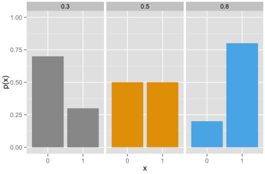

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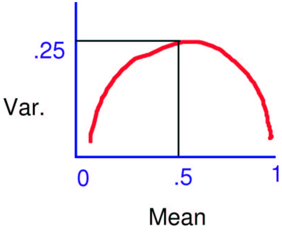

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Discretization for random draws from discrete pmfs

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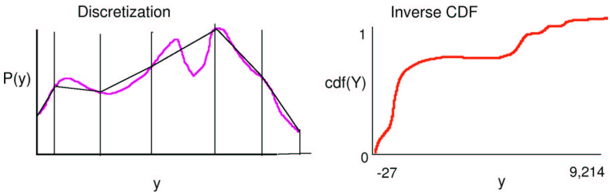

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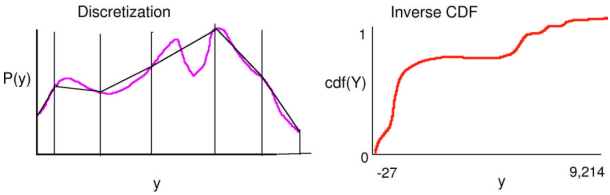

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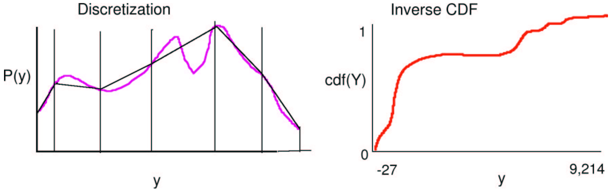

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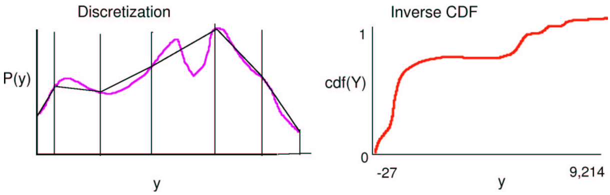

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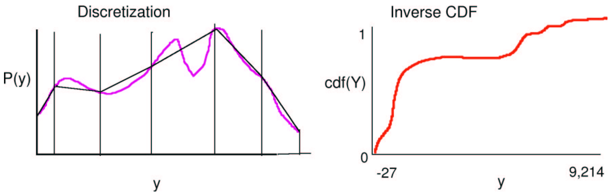

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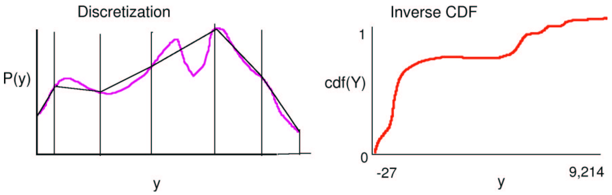

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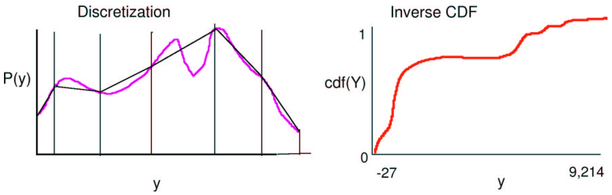

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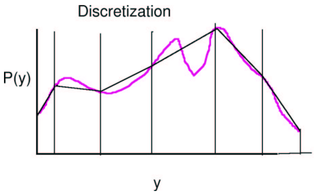
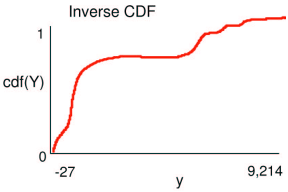
- (Works for a few dimensions, but infeasible for many)

Inverse CDF: drawing from arbitrary continuous pdfs





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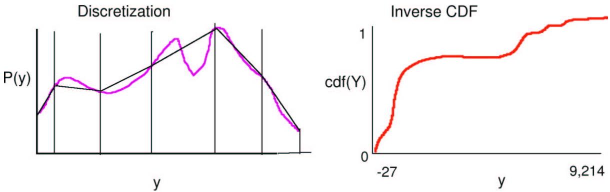

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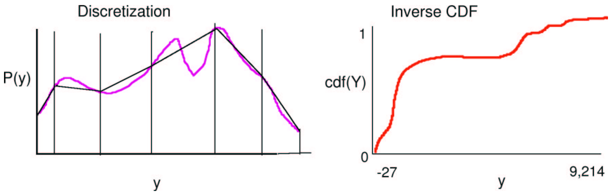

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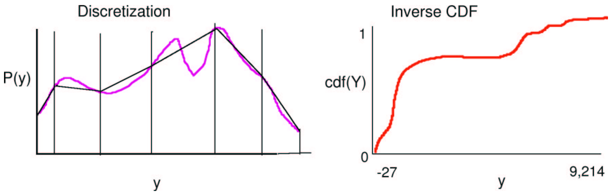

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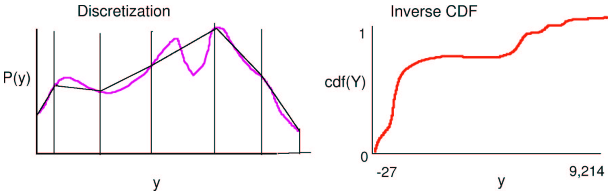

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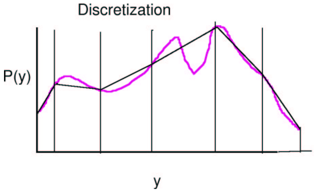
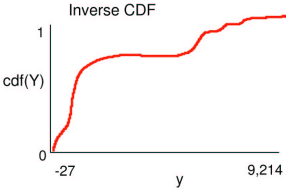
Using Inverse CDF to Improve Discretization Method





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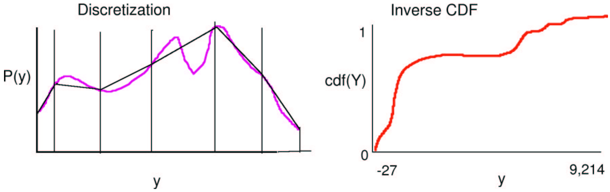

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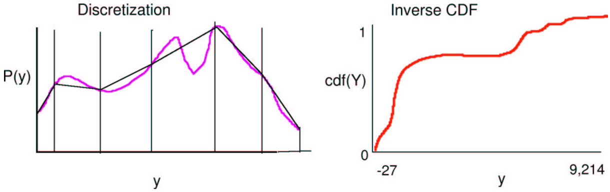

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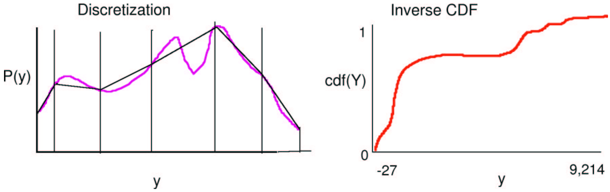

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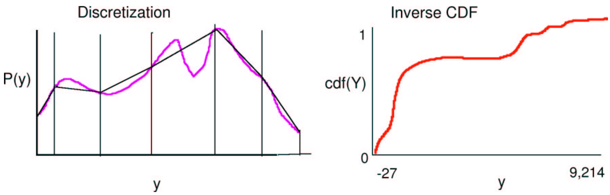

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- Drawing random numbers from arbitrary multivariate densities: now an enormous literature

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Reminder: Equivalent Regression Notation

- Standard version

$$\begin{aligned} Y_i &= x_i\beta + \epsilon_i \qquad &= \text{systematic} + \text{stochastic} \\ \epsilon_i &\sim f_N(0, \sigma^2) \end{aligned}$$

- Alternative version

$$\begin{aligned} Y_i &\sim f_N(\mu_i, \sigma^2) \qquad & \text{stochastic} \\ \mu_i &= x_i\beta & \text{systematic} \end{aligned}$$

- Generalized version

$$\begin{aligned} Y_i &\sim f(\theta_i, \alpha) \qquad & \text{stochastic} \\ \theta_i &= g(x_i, \beta) & \text{systematic} \end{aligned}$$

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- Simulating *once* from this density produces k numbers. Special algorithms are used to generate normal random variates (in R, `mvrnorm()`, from the `MASS` library).

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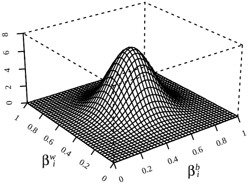
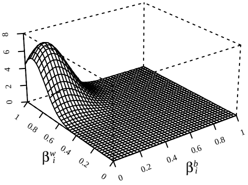
$$N(Y_1|\mu_1, \sigma_1^2) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} N(y_i|\mu_i, \Sigma) dy_2 dy_3 \cdots dy_k$$

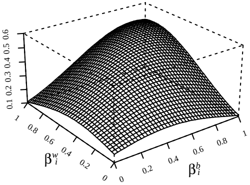
Truncated bivariate normal examples (for β^b and β^w)



(a) 0.5 0.5 0.15 0.15 0



(b) 0.1 0.9 0.15 0.15 0



(c) 0.8 0.8 0.6 0.6 0.5

Parameters are μ_1, μ_2, σ_1, σ_2, and ρ.

Stop here

We will stop here this year and skip to the next set of slides.
Please refer to the slides below for further information on probability densities and random number generation; they offer more sophisticated .

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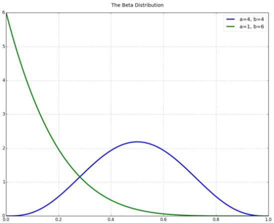

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Result:

$$\text{beta}(y|\mu,\gamma) = \frac{\Gamma\left(\mu\gamma^{-1} + (1-\mu)\gamma^{-1}\right)}{\Gamma\left(\mu\gamma^{-1}\right)\Gamma\left[(1-\mu)\gamma^{-1}\right]} y^{\mu\gamma^{-1}-1}(1-y)^{(1-\mu)\gamma^{-1}-1}$$

Alternative parameterization

Set $\mu = E(Y) = \frac{\alpha}{(\alpha+\beta)}$ and $\frac{\mu(1-\mu)\gamma}{(1+\gamma)} = V(Y) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$, solve for α and β and substitute in.

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Reparameterization like this will be key throughout the course.

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- Add up the $\tilde{z}$'s to get $y = \sum_j^N \tilde{z}_j$, which is a draw from the beta-binomial.

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&= \frac{N!}{y_i!(N-y_i)!} \prod_{j=0}^{y_i-1}(\mu+\gamma j) \prod_{j=0}^{N-y_i-1}(1-\mu+\gamma j) \prod_{j=0}^{N-1}(1+\gamma j)
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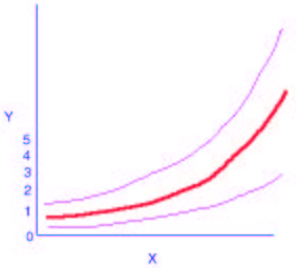

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- How to simulate? We'll use canned random number generators.

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$$\text{gamma}(y|\phi, \sigma^2) = \frac{y^{\phi(\sigma^2-1)^{-1}-1} e^{-y(\sigma^2-1)^{-1}}}{\Gamma[\phi(\sigma^2-1)^{-1}](\sigma^2-1)^{\phi(\sigma^2-1)^{-1}}}$$

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- Draw Y from $\text{Poisson}(y|\tilde{\lambda})$, which gives one draw from the negative binomial.

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\end{aligned}$$