Advanced Quantitative Research Methodology, Lecture Notes: Introduction[1]

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- Some of the best experiences here: getting to know people in other fields

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- Do you have the background for this class? A Test: What's this?

$$b = (X'X)^{-1}X'y$$

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  - We cover different amounts of material each week

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4. Focus, like I will, on learning, not grades: Especially when we work on papers, I will treat you like a colleague, not a student

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  - Come whenever you like; if you can't find me or I'm in a meeting, come back, talk to my assistant in the office next to me, or email any time

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- Most important methods originate *outside* the discipline of statistics (random assignment, experimental design, survey research, machine learning, MCMC methods, . . .). Statistics: abstracts, proves formal properties, generalizes, and distributes results back out.

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- Part of a massive change in the evidence base of the social sciences: (a) surveys, (b) end of period government stats, and (c) one-off studies of people, places, or events $\rightsquigarrow$ numerous new types and huge quantities of (big) data

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- This helps us separate the conventions from underlying statistical theory. (How to get an F in Econometrics: follow advice from Psychometrics. Works in reverse too, even when the foundations are identical.)

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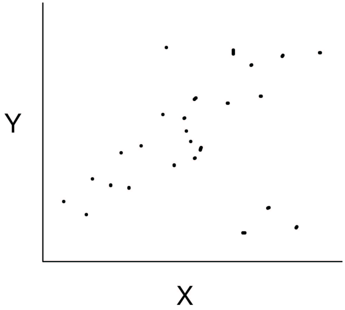

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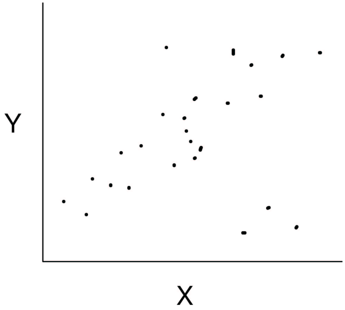

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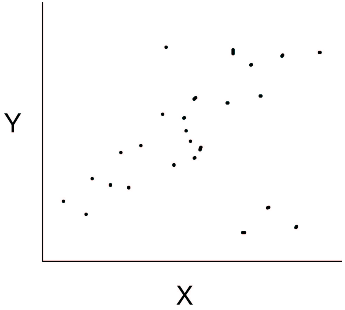

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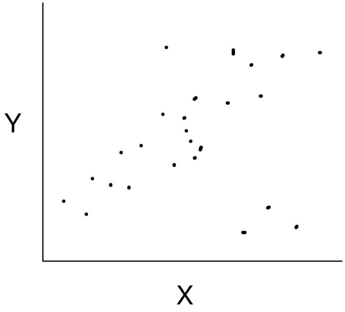

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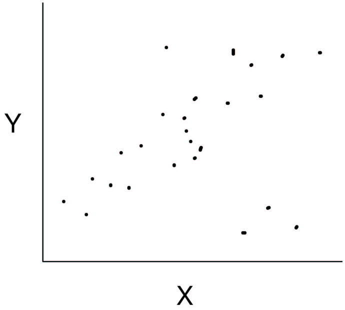

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- from a theory of inference, and for a substantive purpose (like causal estimation, prediction, etc.)

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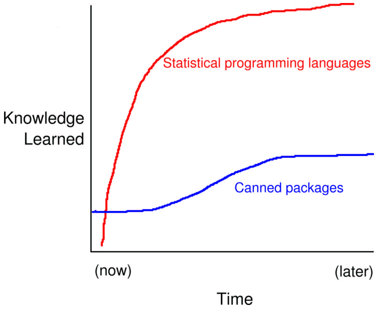

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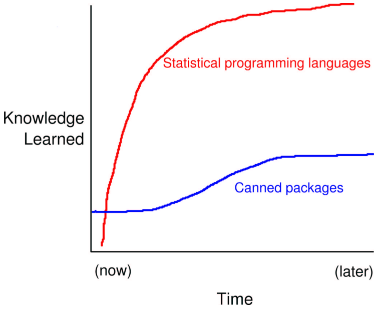

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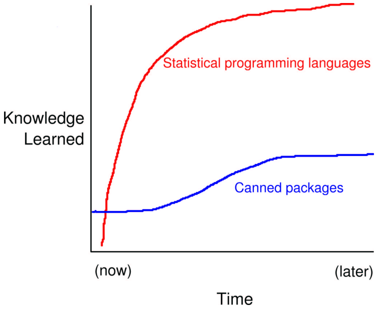

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- and an R program called Zelig (Imai, King, and Lau, 2006-14) which simplifies R and helps you up the steep slope fast (see j.mp/Zelig4)

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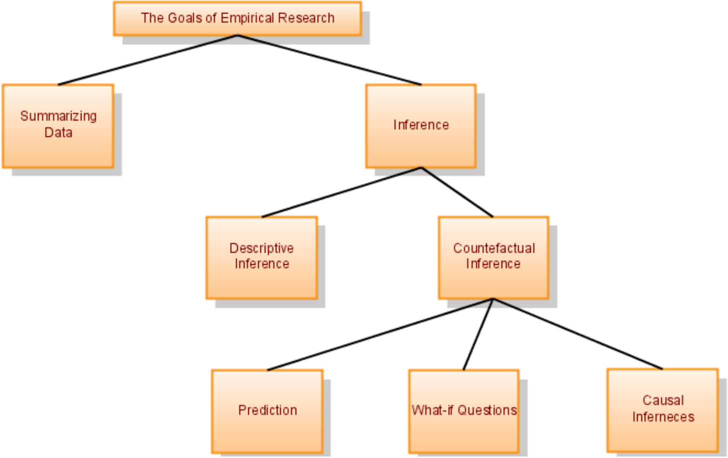

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- Now you know what a model is. (Its an abstraction.)

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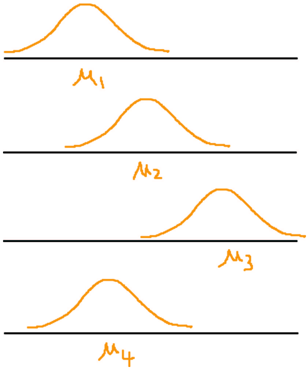

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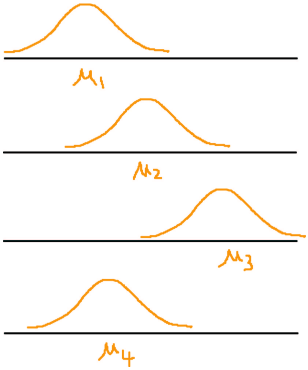

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- (If you know the model, is $R^2 = 1$? Can you predict y perfectly?)

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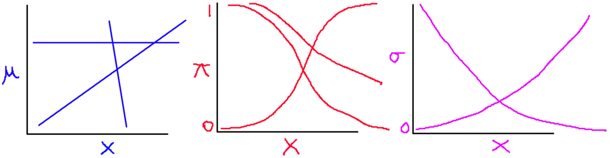

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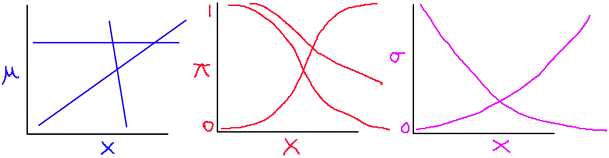

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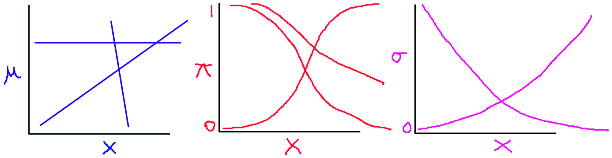

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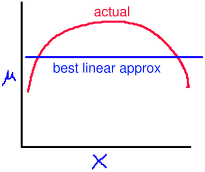

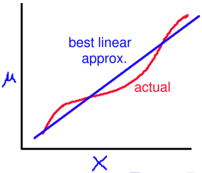

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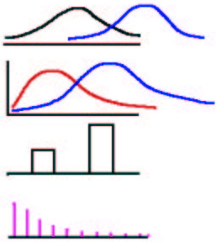

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- Rules can be applied *analytically* or via *simulation*.

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5. $\rightsquigarrow$ Empirical evidence: students get the right answer far more frequently by using simulation than math

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4. Example: estimate the mean of the population	4. Example: Approximate the mean of the distribution

Simulation examples for solving probability problems

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```
Four runs: .538, .550, .547, .524
```

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In Let's Make a Deal, Monte Hall offers what is behind one of three doors. Behind a random door is a car; behind the other two are goats. You choose one door at random. Monte peeks behind the other two doors and opens the one (or one of the two) with the goat. He asks whether you'd like to switch your door with the other door that hasn't been opened yet. Should you switch?

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  if (WinDoor == choice)                      # no switch
  WinNoSwitch <- WinNoSwitch + 1
  doorsLeft <- doors[doors != choice]         # switch
  if (any(doorsLeft == WinDoor))
  WinSwitch <- WinSwitch + 1
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cat("Prob(Car | no switch)=", WinNoSwitch/sims, "\n")
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$\Pr(\text{car}\mid\text{No Switch})$	$\Pr(\text{car}\mid\text{Switch})$
.324	.676
.345	.655
.320	.680
.327	.673

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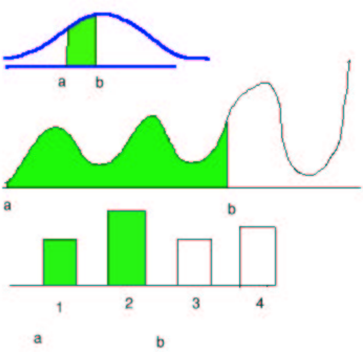
2. $P(Y) \geq 0$ for every Y

Computing Probabilities from Densities



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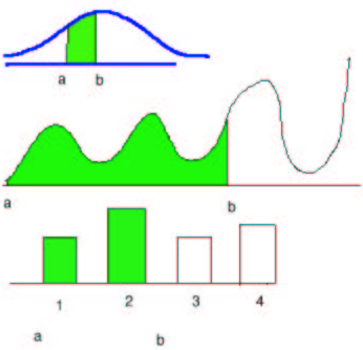

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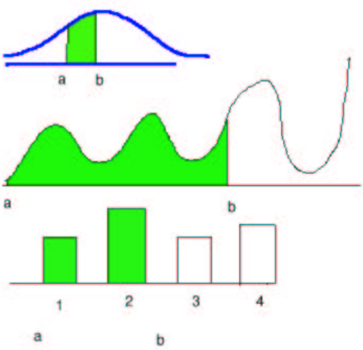

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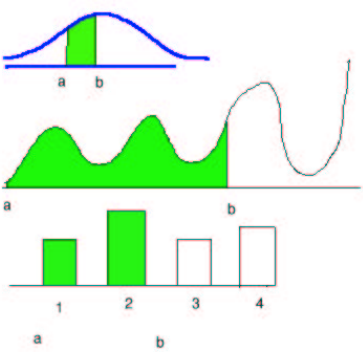

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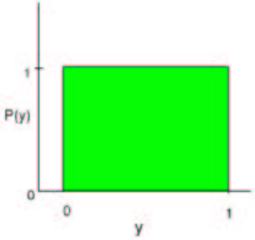

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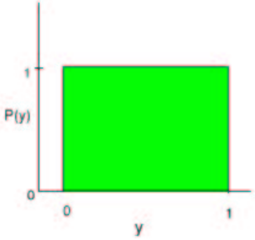

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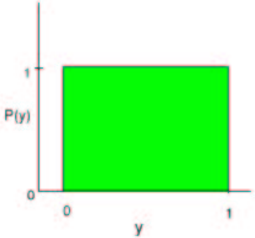

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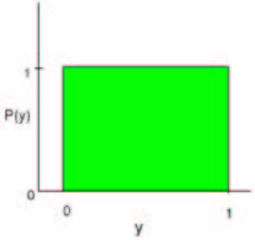

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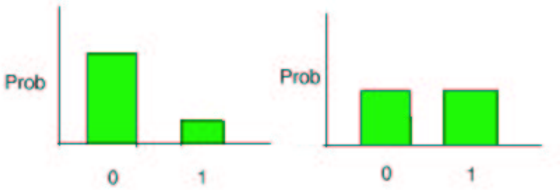
  - $\implies \Pr(Y_i = y|\pi_i) = \pi_i^y(1-\pi_i)^{1-y}$
  - Alternative notation: $\Pr(Y_i = y|\pi_i) = \text{Bernoulli}(y|\pi_i) = f_b(y|\pi_i)$

Graphical summary of the Bernoulli



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Variance of the Bernoulli (uses above results)

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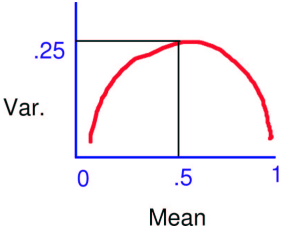

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“Discretization” for random draws from discrete pmfs, given uniform random numbers

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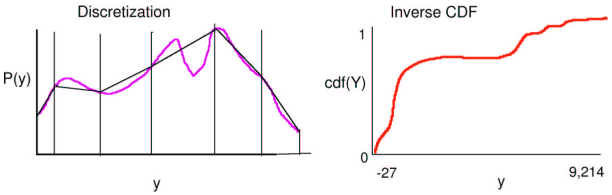

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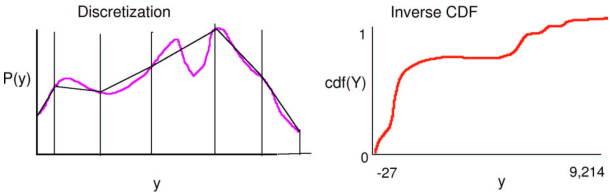

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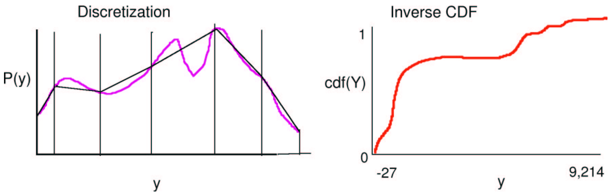

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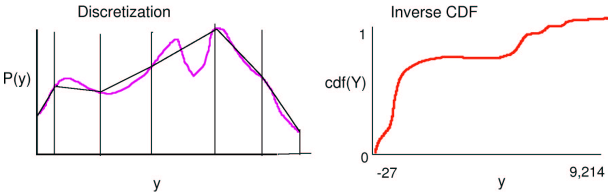

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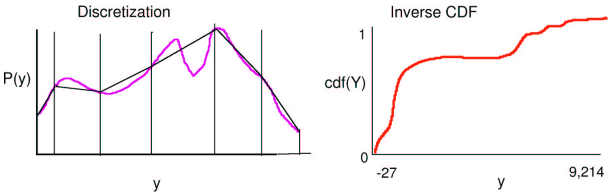

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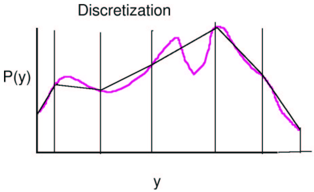
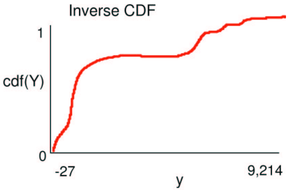
Inverse CDF method for random draws from continuous pdfs given uniform random numbers





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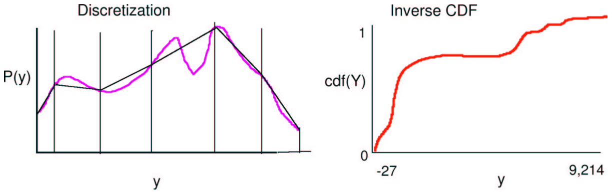

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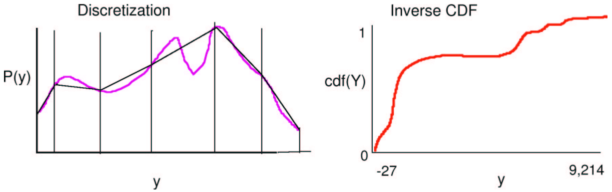

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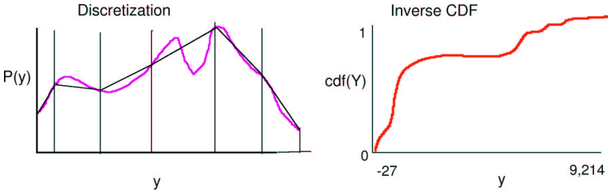

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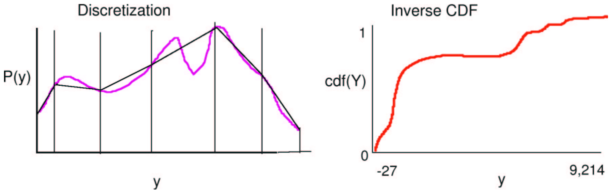

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- Then $F^{-1}(U)$ gives a random draw from $f(Y)$.

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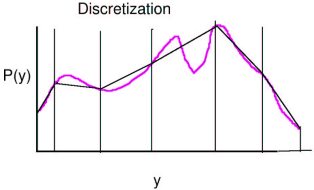

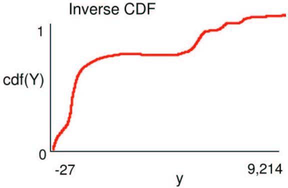

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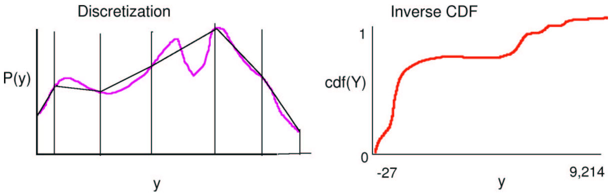

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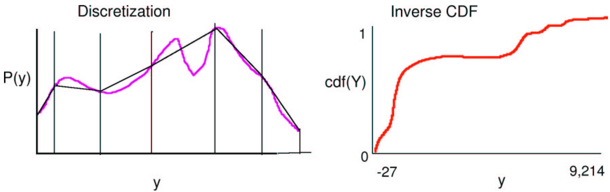

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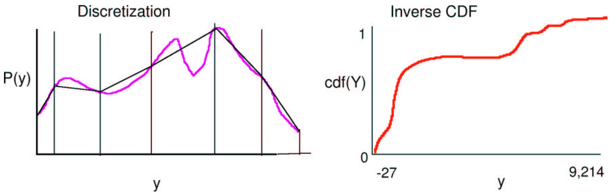

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- Drawing random numbers from arbitrary multivariate densities: now an enormous literature

Stop here

We will stop here this year and skip to the next set of slides. Please refer to the notes below for further information on probability densities and random number generation.

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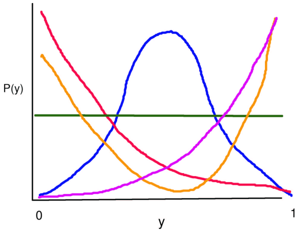

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Reparameterization like this will be key throughout the course.

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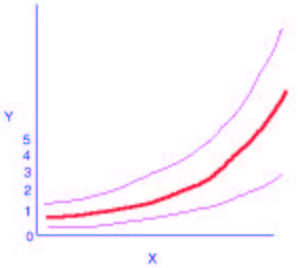

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$$\text{gamma}(y|\phi, \sigma^2) = \frac{y^{\phi(\sigma^2-1)^{-1}-1} e^{-y(\sigma^2-1)^{-1}}}{\Gamma[\phi(\sigma^2-1)^{-1}](\sigma^2-1)^{\phi(\sigma^2-1)^{-1}}}$$

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&= \frac{\Gamma\left(\frac{\phi}{\sigma^2-1} + y_i\right)}{y_i! \Gamma\left(\frac{\phi}{\sigma^2-1}\right)} \left(\frac{\sigma^2 - 1}{\sigma^2}\right)^{y_i} \left(\sigma^2\right)^{\frac{-\phi}{\sigma^2-1}}
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- Simulating *once* from this density produces k numbers. Special algorithms are used to generate normal random variates (in R, `mvrnorm()`, from the `MASS` library).

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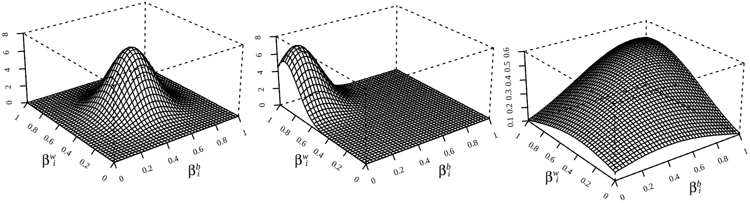
Truncated bivariate normal examples (for β^b and β^w)



(a) 0.5 0.5 0.15 0.15 0 (b) 0.1 0.9 0.15 0.15 0 (c) 0.8 0.8 0.6 0.6 0.5

Parameters are μ_1, μ_2, σ_1, σ_2, and ρ.

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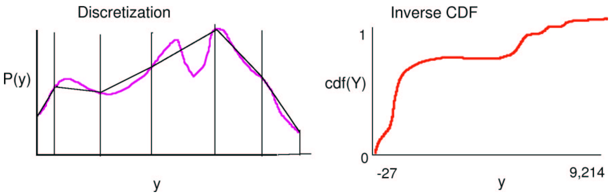

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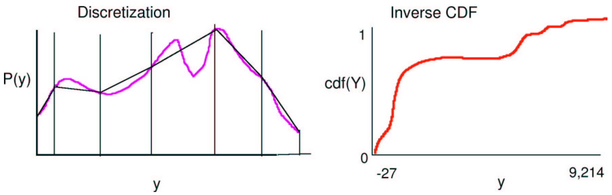

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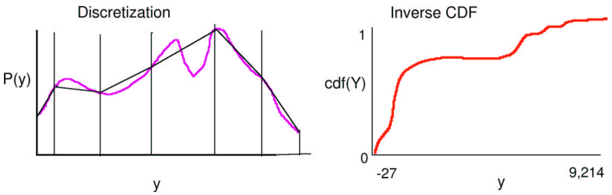

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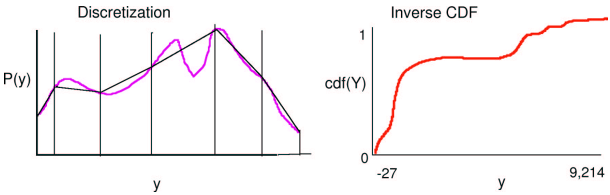
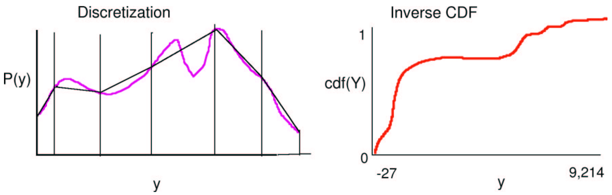
- Compute by linear approximation area (density/probability) in each interval

"Discretization" for random draws from discrete pmfs, given uniform random numbers



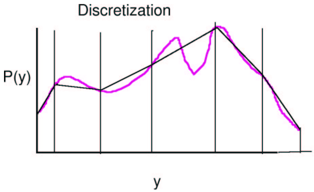
- Divide up PDF into a grid
- Compute by linear approximation area (density/probability) in each interval
- Map [0,1] to the densities proprtionally

"Discretization" for random draws from discrete pmfs, given uniform random numbers

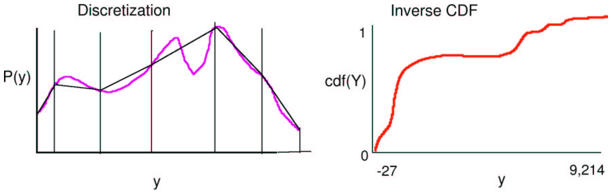


- Divide up PDF into a grid
- Compute by linear approximation area (density/probability) in each interval
- Map [0,1] to the densities proprtionally
- Not feasible for multivariate random number generation

Inverse CDF method for random draws from continuous pdfs given uniform random numbers

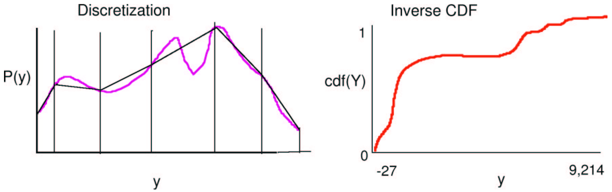


Inverse CDF method for random draws from continuous pdfs given uniform random numbers



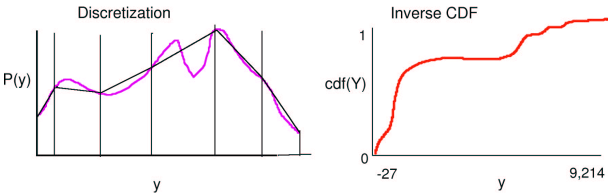
- From the pdf $f(Y)$, compute the cdf:
  $\Pr(Y \leq y) \equiv F(y) = \int_{-\infty}^{y} f(z)dz$

Inverse CDF method for random draws from continuous pdfs given uniform random numbers



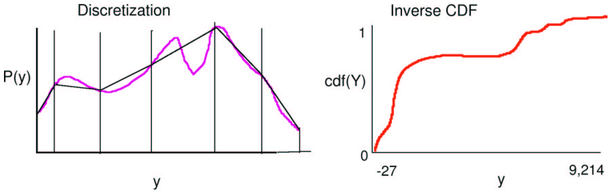
- From the pdf $f(Y)$, compute the cdf: $\Pr(Y \leq y) \equiv F(y) = \int_{-\infty}^{y} f(z)dz$
- Define the inverse cdf $F^{-1}(y)$, such that $F^{-1}[F(y)] = y$

Inverse CDF method for random draws from continuous pdfs given uniform random numbers



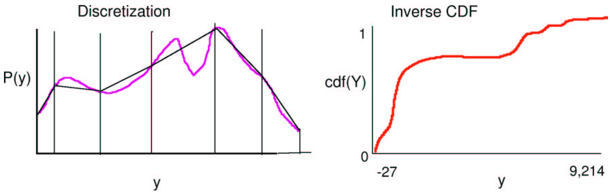
- From the pdf $f(Y)$, compute the cdf: $\Pr(Y \leq y) \equiv F(y) = \int_{-\infty}^{y} f(z)dz$
- Define the inverse cdf $F^{-1}(y)$, such that $F^{-1}[F(y)] = y$
- Draw random uniform number, U

Inverse CDF method for random draws from continuous pdfs given uniform random numbers

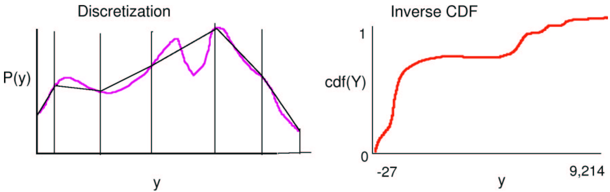


- From the pdf $f(Y)$, compute the cdf: $\Pr(Y \leq y) \equiv F(y) = \int_{-\infty}^{y} f(z)dz$
- Define the inverse cdf $F^{-1}(y)$, such that $F^{-1}[F(y)] = y$
- Draw random uniform number, U
- Then $F^{-1}(U)$ gives a random draw from $f(Y)$.

A Refined Discretization method

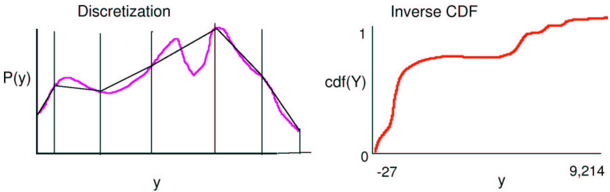


A Refined Discretization method



- Choose interval randomly as above

A Refined Discretization method



- Choose interval randomly as above
- Draw a number within an interval by the inverse CDF method applied to the trapezoidal approximation.