Quantitative Social Science Methods, I, Lecture Notes: Multiple Equation Models

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Identification

Seemingly Unrelated Regression Models

Reciprocal Causation (Endogeneity)

Multinomial Choice Models

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- Reading: *Unifying Political Methodology*, Chapter 8

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&= x_{1i}\beta_1 + x_{2i}(\beta_2 + \beta_3)
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- Likelihood of $\{\beta_2 = 3, \beta_3 = 5\}$ is same as $\{\beta_2 = 5, \beta_3 = 3\}$

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- Joint Stochastic Component

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  - Not an assumption about relationships among $Y_{i1}, \ldots, Y_{iN}$

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  - $\rightsquigarrow$ Equivalent to maximizing equation-by-equation likelihoods

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- Computational tricks: can make estimation a lot faster

Identification

Seemingly Unrelated Regression Models

Reciprocal Causation (Endogeneity)

Multinomial Choice Models

Model for Reciprocal Causation

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- What do we do with $\mu_i = (\mu_{i1}, \mu_{i2})$?

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  - My advice: wherever possible, choose a different project

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Reciprocal Causation (Endogeneity)

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- The Likelihood: $L(\beta|y) = \prod_{i=1}^{n} \left[\prod_{j=1}^{J} \pi_{ij}^{y_{ij}} \right]$

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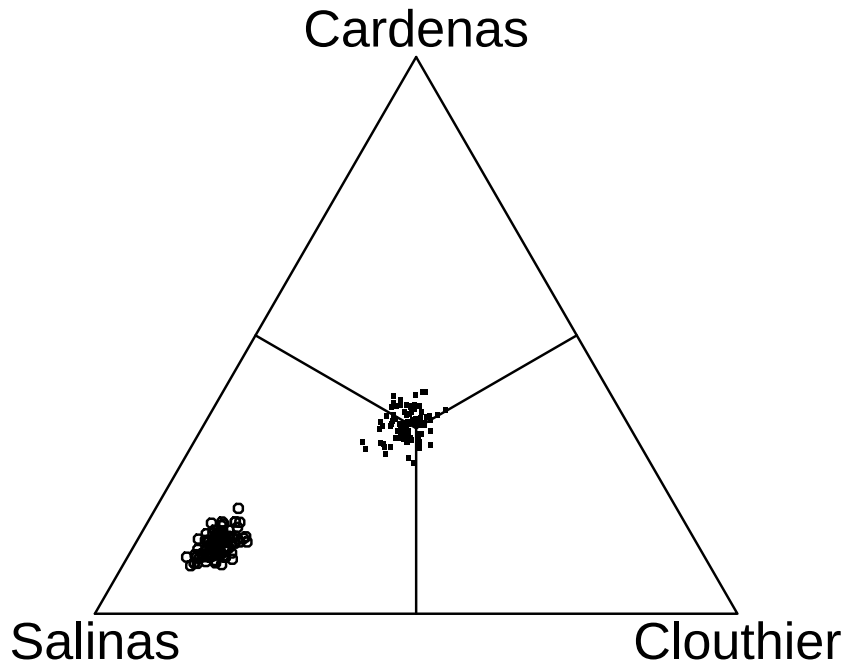
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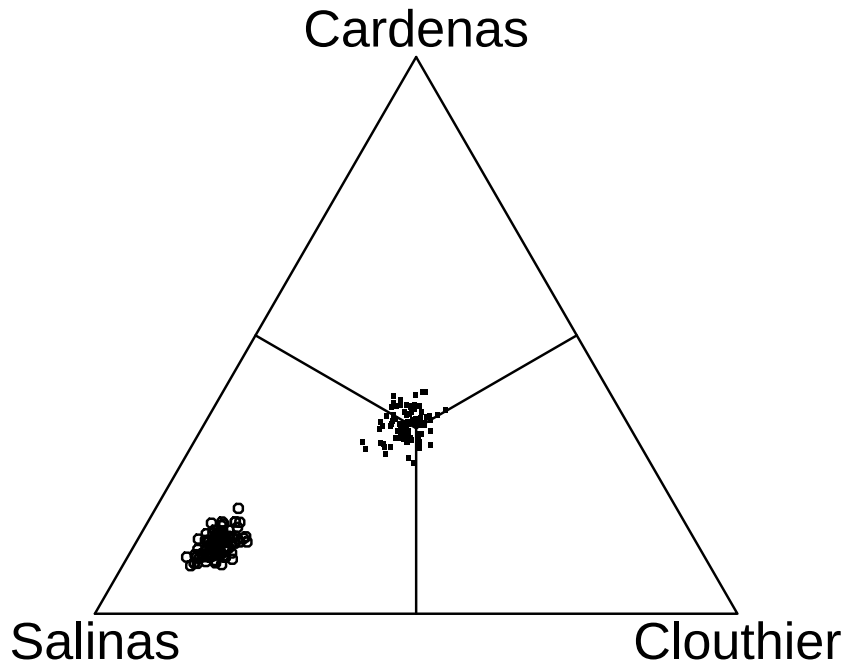
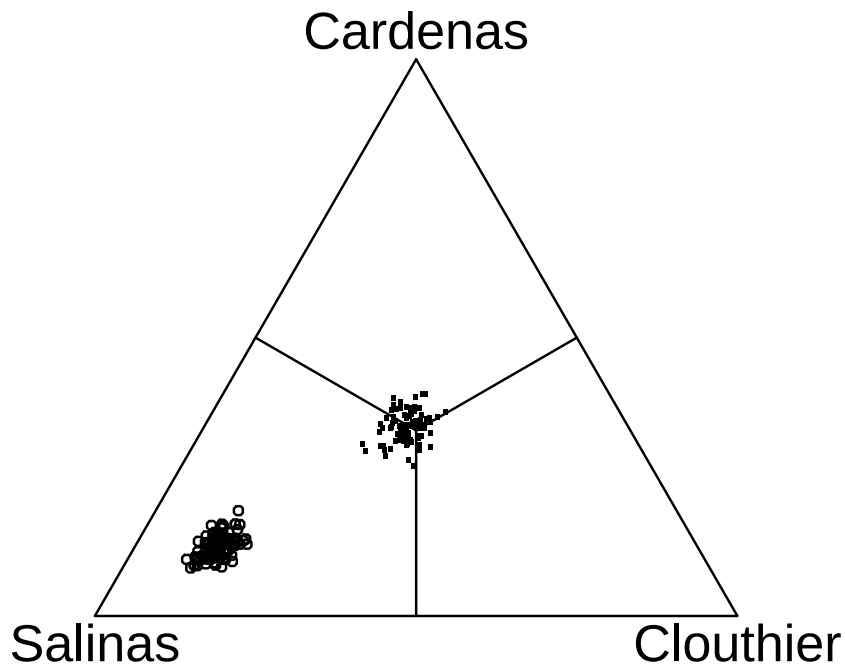


- Each point: an election outcome where all voters believe Salinas' PRI is strengthing (for the "o"'s in the bottom left) or weakening (for the "."'s in the middle), with other variables held constant at their means.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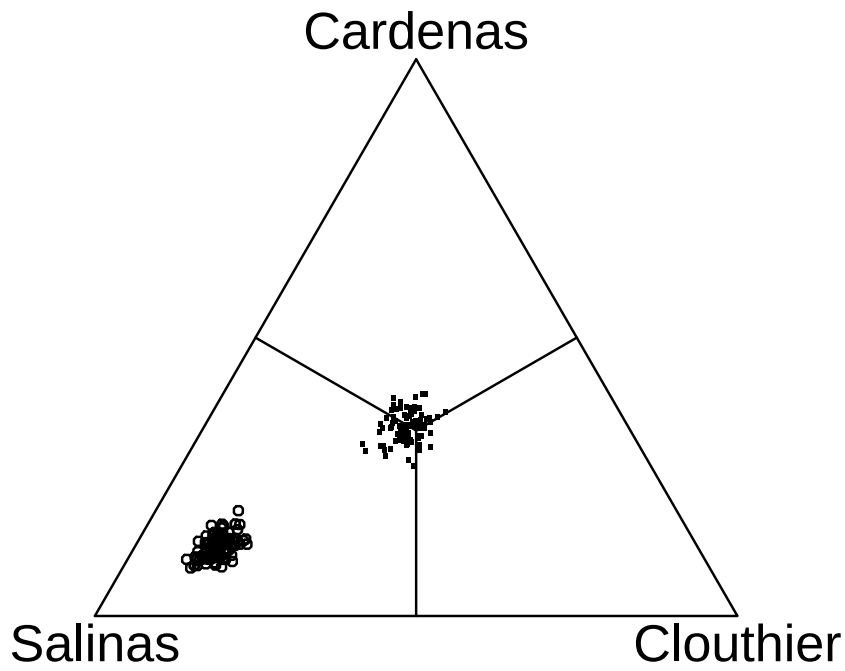

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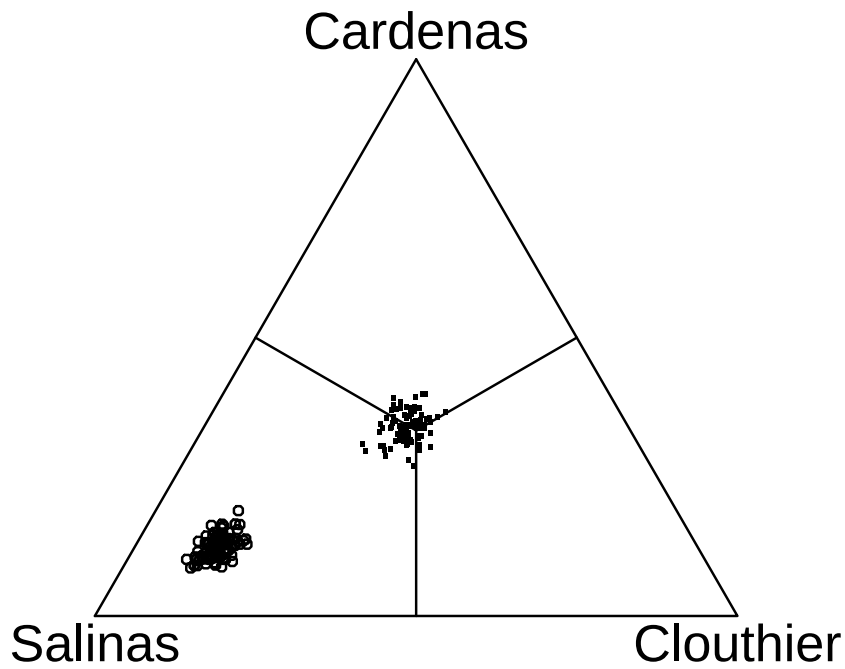

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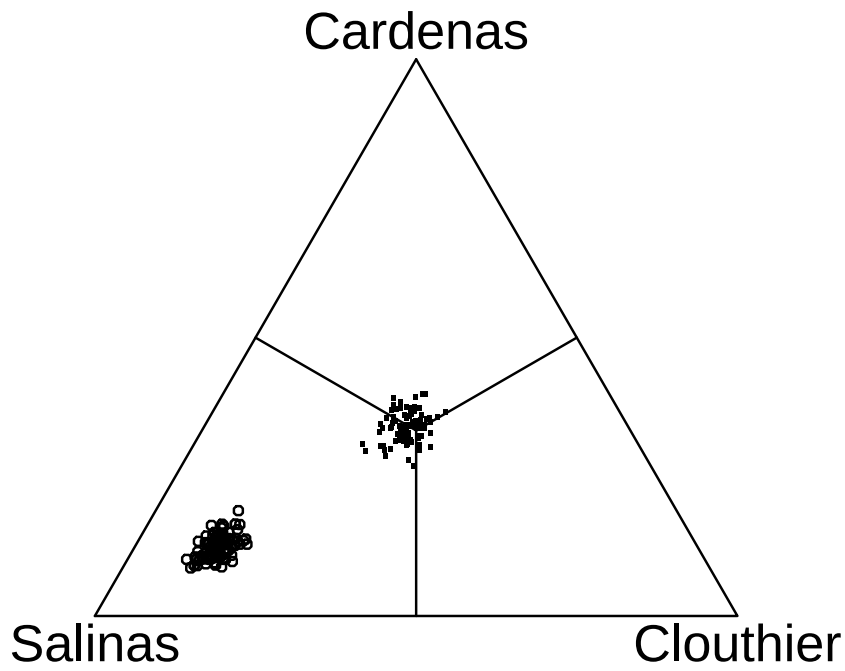

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- If voters believe PRI is weakening, its a tossup.
- $\rightsquigarrow$ Despite voter fraud, Salinas probably did defeat a divided opposition in 1988. The PRI lost the next election (finally, after 72 years in power)