Quantitative Social Science Methods, I, Lecture Notes: Discrete and Limited Outcome Models

Gary King[1]
Institute for Quantitative Social Science
Harvard University

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[1]GaryKing.org

Ordered Dependent Variable Models

Grouped Binary Variable Models

Count Models

Duration Models and Censoring

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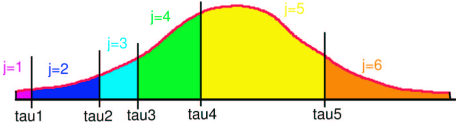

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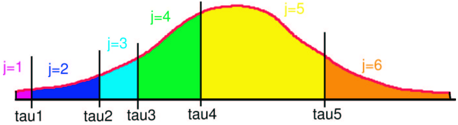

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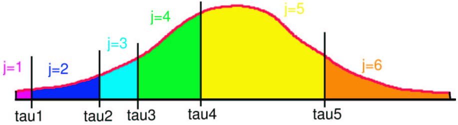

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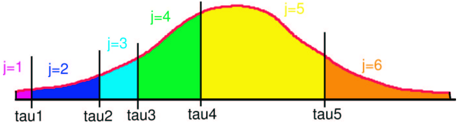

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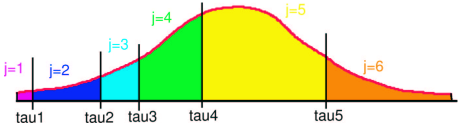

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- If instead $Y_i^* \sim \text{STL}(y_i^*|\mu_i)$: ordered logit model

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- Careful of optimization constraints: $\tau_{j-1} < \tau_j, \quad \forall j$

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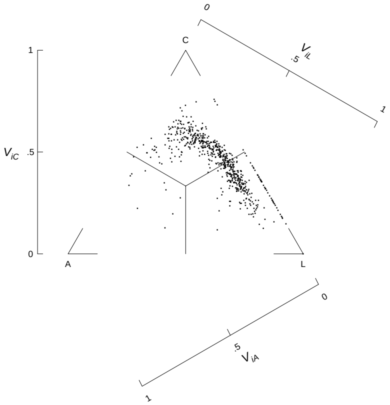

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- Compute QOIs: mean, SD, CI's, histogram, etc.

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- Compute QOIs: mean, SD, CI's, histogram, etc.

Ordered Dependent Variable Models

Grouped Binary Variable Models

Count Models

Duration Models and Censoring

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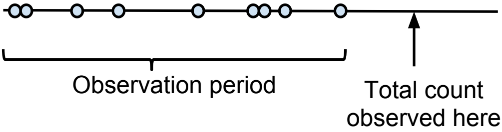

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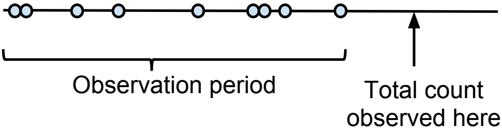

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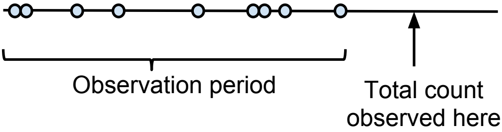

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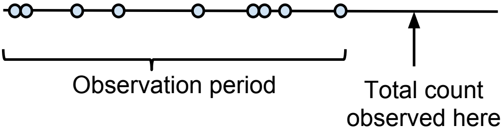

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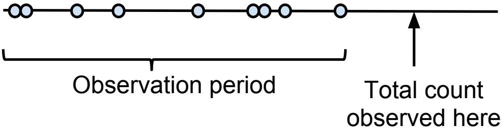

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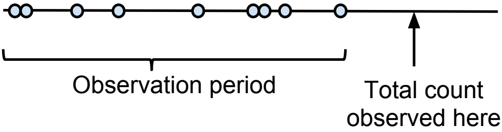

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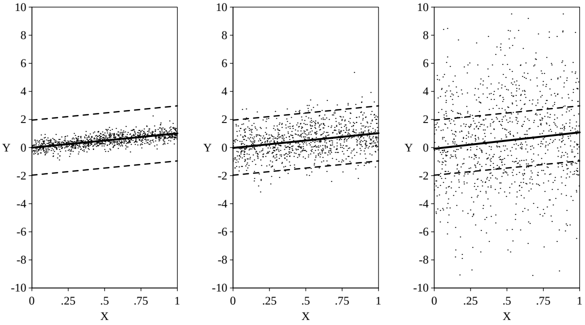
  - Variance conditional on X, dispersion changes with specification

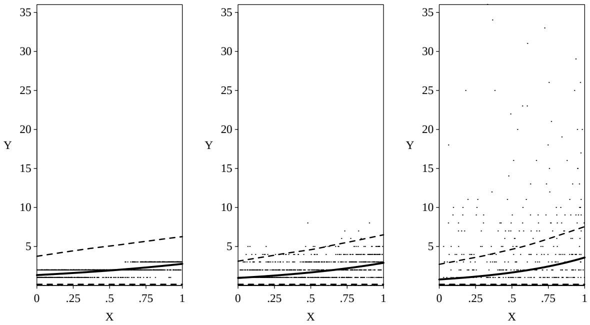
Problems without an extra parameter? Stylized Normal



(a) (b) (c)

$E(Y|X)$ and 95% CI.

Problems without an extra parameter? Poisson



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Negative Binomial Event Count Model

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  - Careful of off-the-shelf programs: maybe $V(Y|X) = \phi(1 + \sigma^2\phi)$

A Generalized Event Count (GEC) Model

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An event count model with under-, Poisson, and over-dispersion

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Crazy math, but same logical structure as the other models today!

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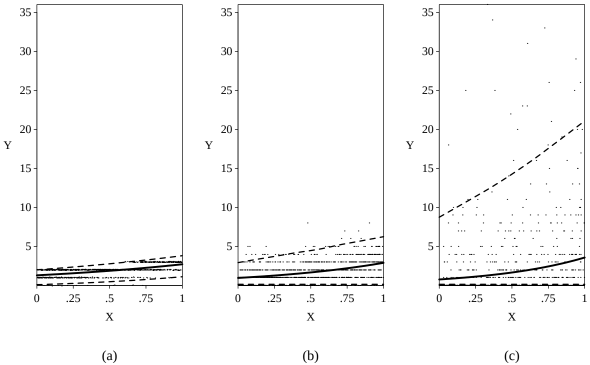
- References: King and Signorino, *Political Analysis*, 1996. King *American Journal of Political Science*, 1989

What happens with the extra parameter? GEC



$E(Y|X)$ and 95% CI.

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Duration Models and Censoring

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\begin{aligned}
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&= \sum_{i=1}^{n} \left\{-X_i\beta - e^{-X_i\beta} y_i\right\}
\end{aligned}
$$

The Exponential Model

- The density: Same first principles as the Poisson, except that we observe the duration between events.
- The Model:
  $Y_i \sim \text{expon}(\lambda_i) = \lambda_i e^{-\lambda_i y_i}, \quad E(Y_i) \equiv \frac{1}{\lambda_i} = \frac{1}{e^{-x_i\beta}} = e^{x_i\beta}$
- Quiz: When would you apply this model?
- Log-likelihood

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- Reference: King, Alt, Burns, Laver, "A Unified Model of Cabinet Dissolution in Parliamentary Democracies," *American Journal of Political Science*, 1990

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  - Better: Include censoring information in the likelihood

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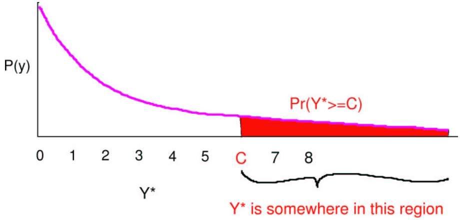

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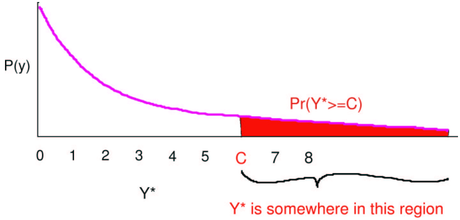

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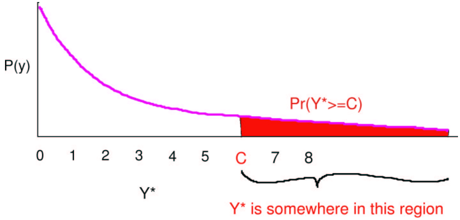

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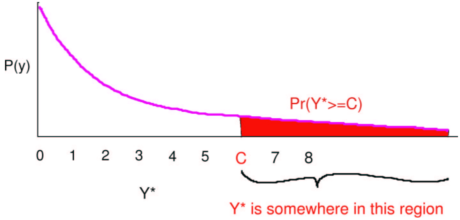

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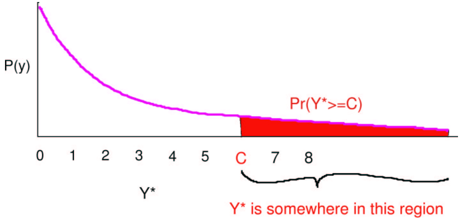

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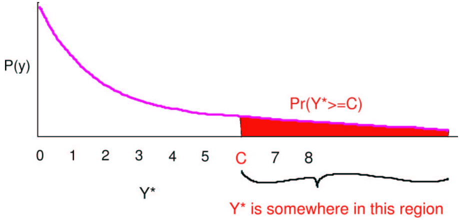

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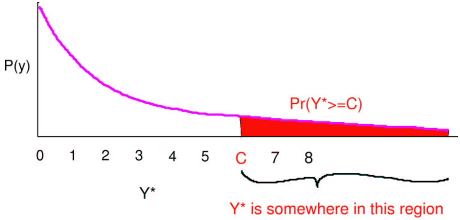

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$$L(\beta|y) = \left[\prod_{y_i^* < C} \text{expon}(y_i|\lambda_i)\right]\left[\prod_{y_i^* \geq C} \Pr(Y_i^* \geq C)\right]$$