Quantitative Social Science Methods, I, Lecture Notes: Inference

Gary King[1]
Institute for Quantitative Social Science
Harvard University

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[1] GaryKing.org

How to fit a line to a scatterplot?

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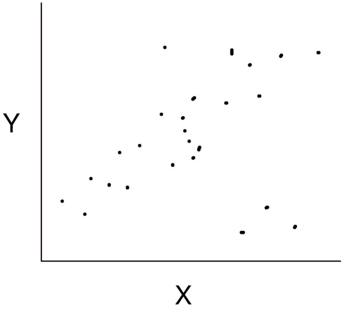

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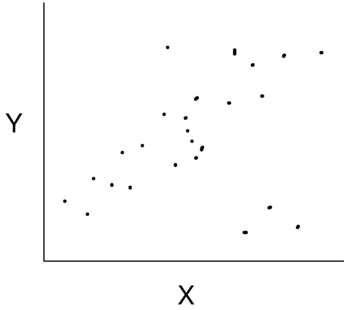

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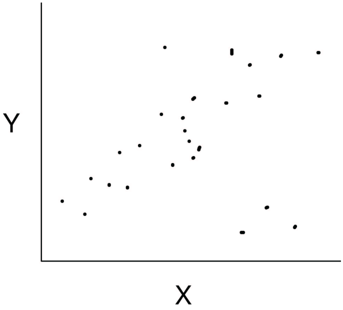

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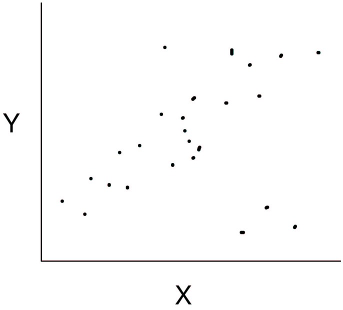

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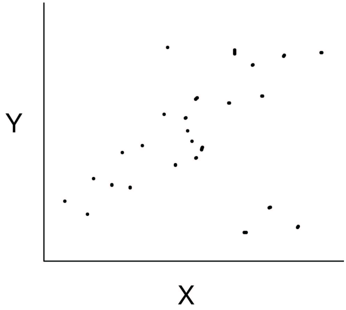

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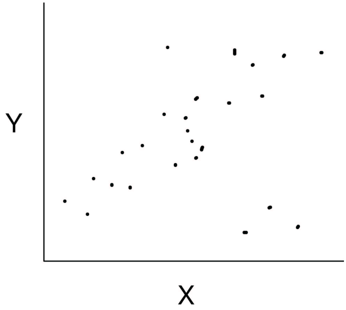

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- (It's a pretty dumb question, don't you think?)

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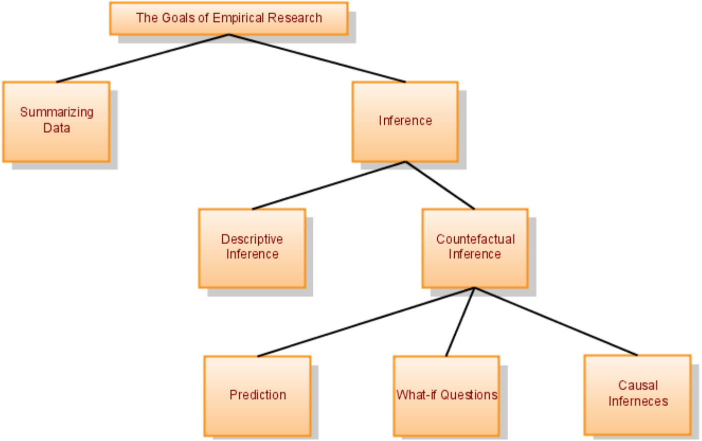

The Problem of Inference

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- A more reasonable, limited goal. Let $M = \{M^*, \theta\}$, where M^* is assumed & θ is to be estimated:

$$P(\theta \mid y, M^*) \equiv P(\theta|y)$$

The Impossibility of Inference Without Assumptions

Three Theories of Inference: Overview

Likelihood: Example, Derivation, Properties

Uncertainty in Likelihood Inference

Simulation from Likelihood Models

Extending the Linear Model with a Variance Function

Building Theories of Inference

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- The two differ on the rest

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- The likelihood principle: the data y only affect inferences through the likelihood function, $L(\theta|y) = k(y)P(y|\theta)$

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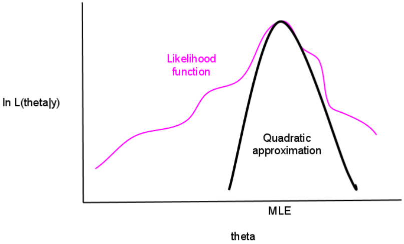

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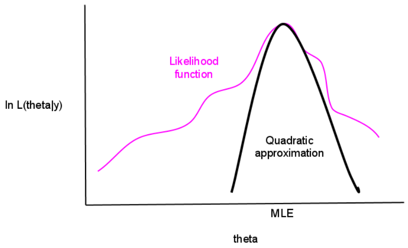

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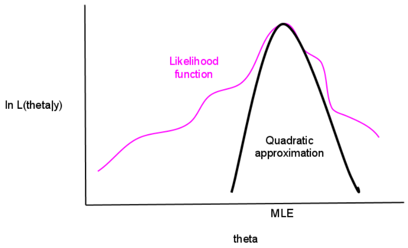

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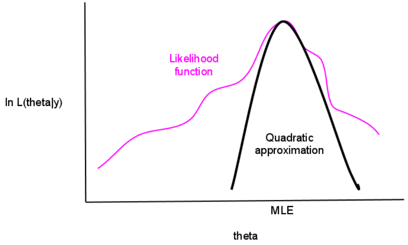

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- $P(\theta)$, the prior density —
  the way Bayes differs from likelihood

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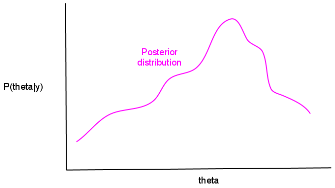
4. If variables A and B are both unknown, then the distribution of A alone is $P(A) = \int \mathrm{P}(A, B) dB = \int P(A|B)P(B) dB$.

The posterior density, P($\theta|y$)



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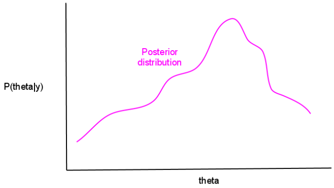

- Like L, it's a summary estimator

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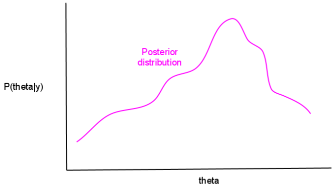

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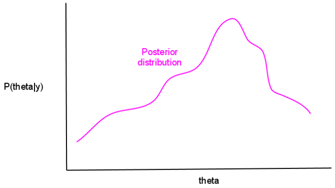

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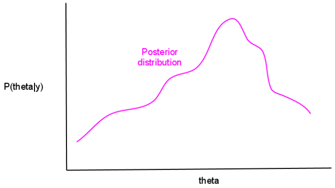

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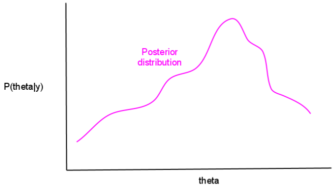

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- Few fights now between Bayesians and likelihoodists

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- Once discredited; making a comeback through the preregistration movement

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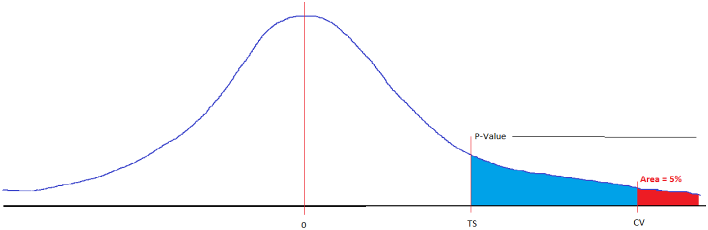

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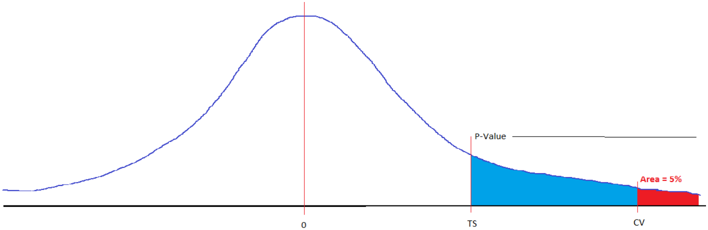

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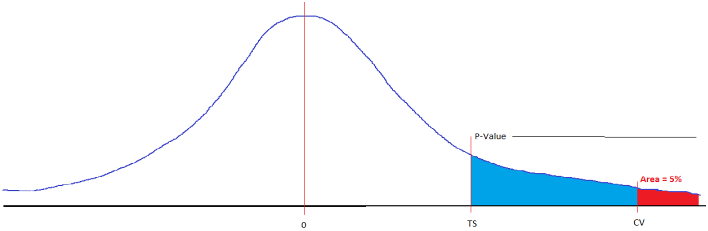

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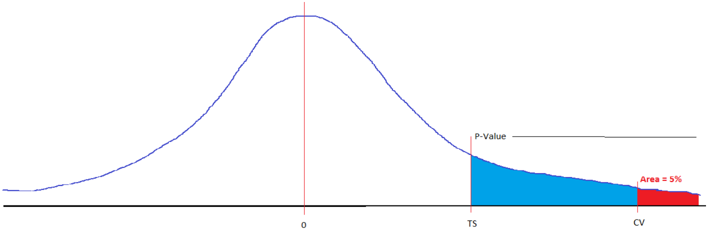

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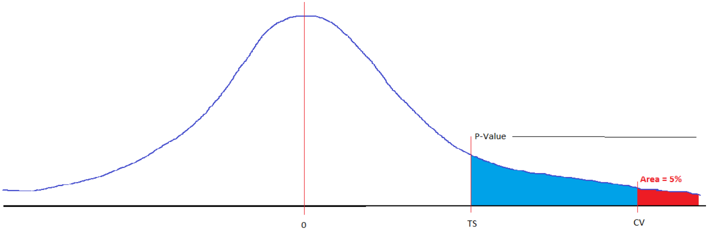

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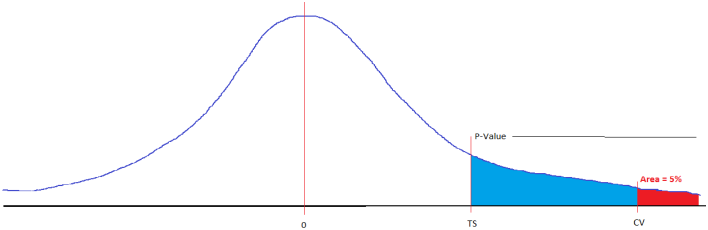

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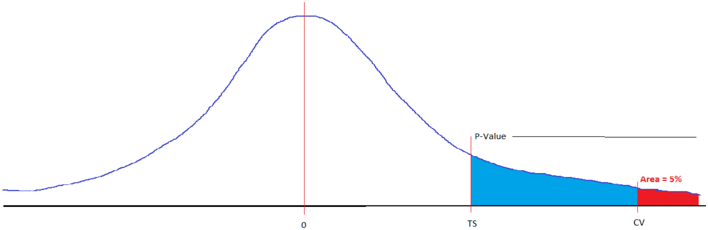

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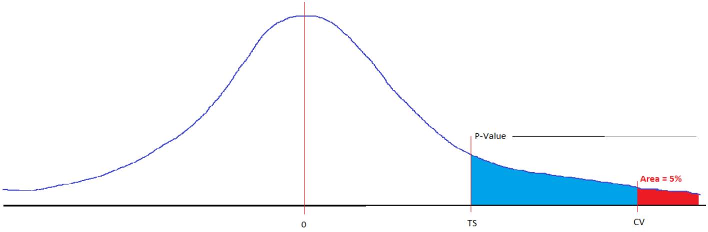

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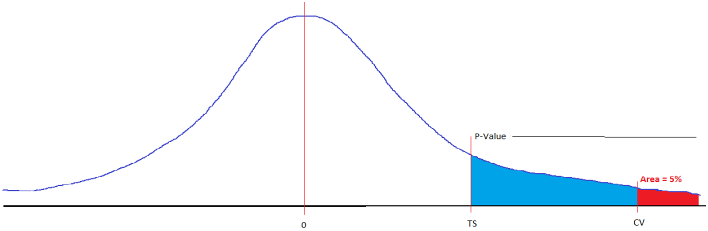

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- $\rightsquigarrow$ Can use likelihood: to compute tests and p-values.

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4. Methods for applied researchers: either useful or irrelevant

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  - Bayesian model averaging, with a large enough class of models to average over
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  - Models with highly flexible functional forms
- The key: No assumptions can be trusted; all theories of inference condition on assumptions and so data analysts always struggle trying to understand and get around them

The Impossibility of Inference Without Assumptions

Three Theories of Inference: Overview

Likelihood: Example, Derivation, Properties

Uncertainty in Likelihood Inference

Simulation from Likelihood Models

Extending the Linear Model with a Variance Function

A Simple Likelihood Model: Stylized Normal, no X

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Quiz: What can you do with this probability density?

Derive the Log-Likelihood

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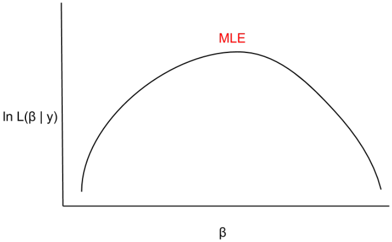
Quiz: What subs for β to make $\ln L$ the largest? What's that called?

Log-likelihood interpretation



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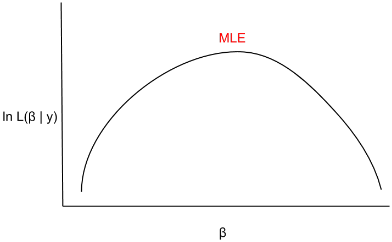

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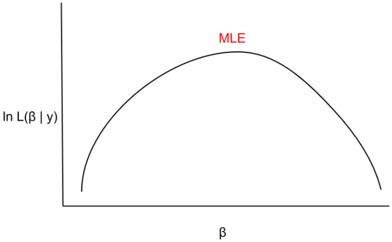

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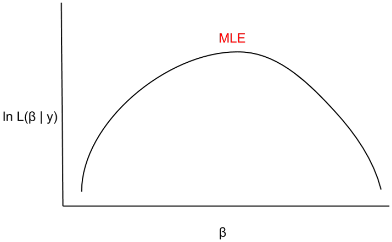

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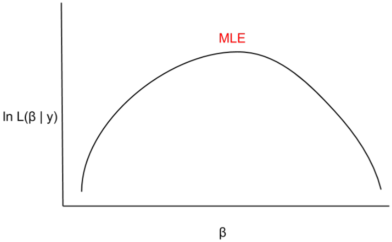

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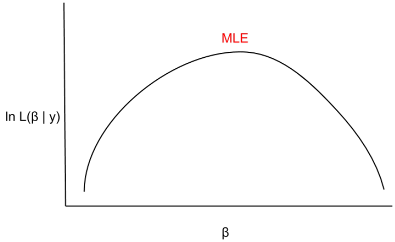

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5. No reason to summarize this curve with only the MLE

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   - In fact, it's a great idea! (e.g., King, Schneer, White 2017)

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Sampling distributions of the MLE: CLT vs LLN

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$\rightsquigarrow$ Why asymptotic approximations may work in small samples

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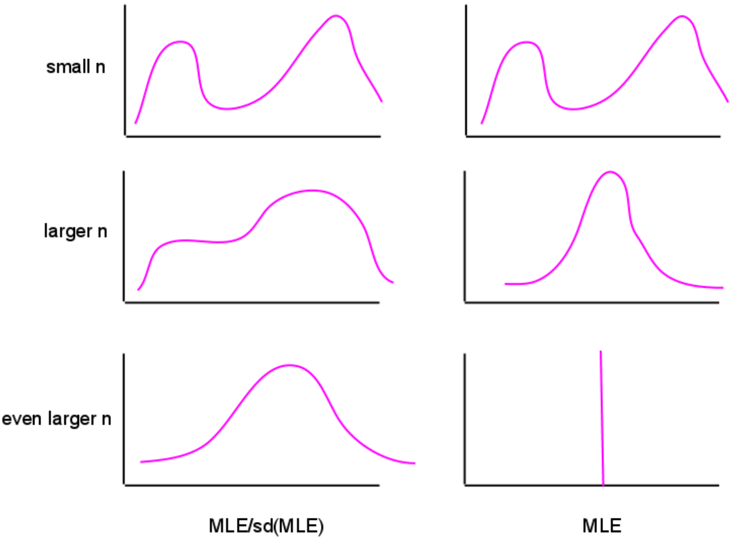

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Three Measures of Uncertainty

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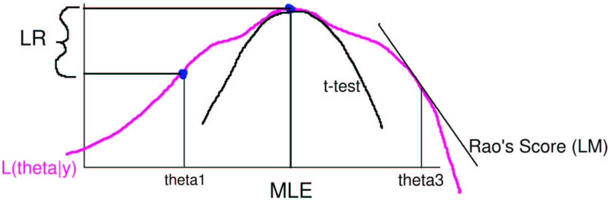

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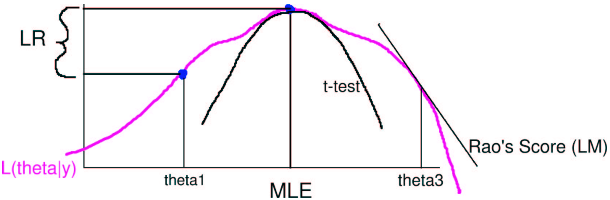

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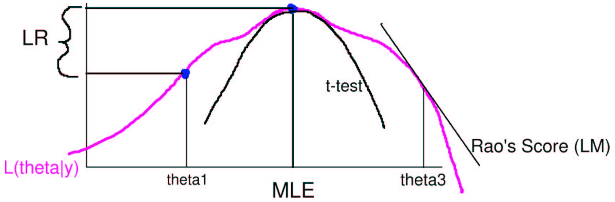

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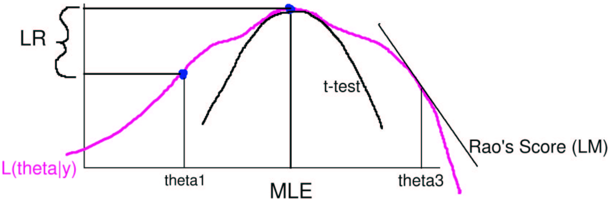

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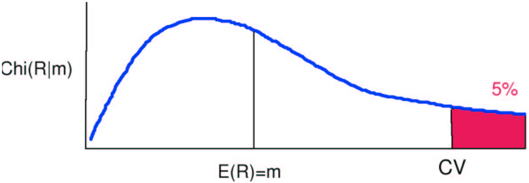

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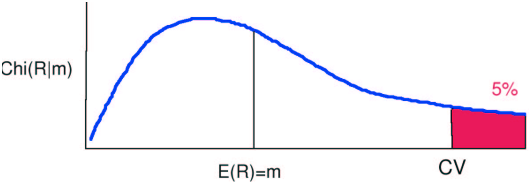

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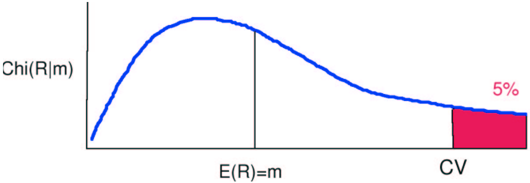

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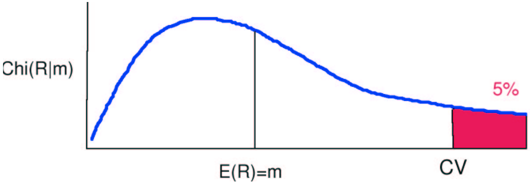

- If restrictions have no effect: $E(R) = m$.
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Uncertainty via Standard Errors

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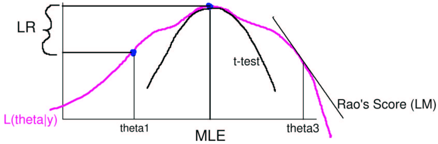

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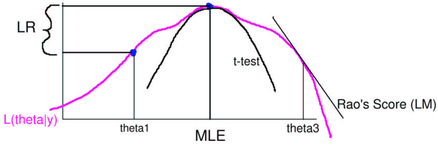

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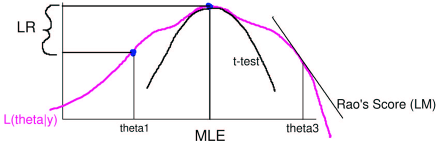

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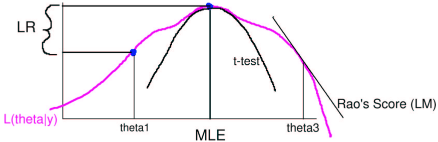

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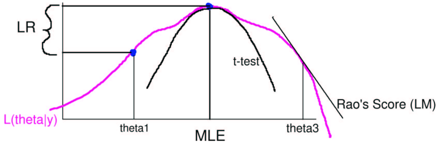

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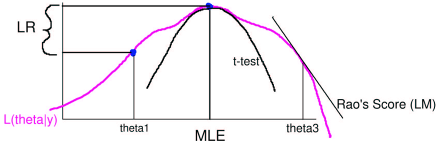

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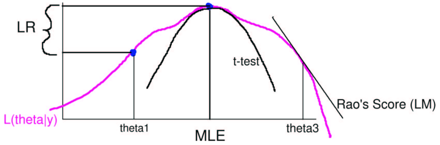

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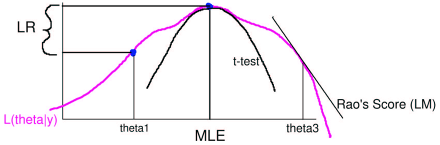

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Second derivative: coefficient c on squared term for any model

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- Quiz: What's the QOI? Is it θ?

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Quiz: What are this model's weaknesses?

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&= -\frac{1}{2}\sum_{i=1}^{n}\sum_{t=1}^{T} \left[\ln(2\pi) + \ln\sigma^2 + \frac{(y_{it}-\mu_{it})^2}{\sigma^2}\right]
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\ln L(\beta, \sigma^2 | y) &= \ln \left[\prod_{i=1}^{n} \prod_{t=1}^{T} L(\mu_{it}, \sigma^2 | y_{it}) \right] = \sum_{i=1}^{n} \sum_{t=1}^{T} \ln L(y_{it} | \mu_{it}, \sigma^2) \\
&= \sum_{i=1}^{n} \sum_{t=1}^{T} \ln \left[(2\pi\sigma^2)^{-1/2} e^{\frac{-(y_{it}-\mu_{it})^2}{2\sigma^2}} \right] \\
&= \sum_{i=1}^{n} \sum_{t=1}^{T} \left[-\frac{1}{2} \ln(2\pi\sigma^2) - \frac{(y_{it}-\mu_{it})^2}{2\sigma^2} \right] \\
&= -\frac{1}{2} \sum_{i=1}^{n} \sum_{t=1}^{T} \left[\ln(2\pi) + \ln\sigma^2 + \frac{(y_{it}-\mu_{it})^2}{\sigma^2} \right] \\
&\doteq -\frac{1}{2} \sum_{i=1}^{n} \sum_{t=1}^{T} \left[\ln\sigma^2 + \frac{(y_{it}-\mu_{it})^2}{\sigma^2} \right]
\end{aligned}
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Log-Likelihood

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&= \sum_{i=1}^{n}\sum_{t=1}^{T} \ln\left[(2\pi\sigma^2)^{-1/2} e^{\frac{-(y_{it}-\mu_{it})^2}{2\sigma^2}}\right] \\
&= \sum_{i=1}^{n}\sum_{t=1}^{T} \left[-\frac{1}{2}\ln(2\pi\sigma^2) - \frac{(y_{it}-\mu_{it})^2}{2\sigma^2}\right] \\
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\end{aligned}
$$

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R Code for the Log-Likelihood

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- (Recall) mathematical Form:

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- An R function:

```
loglik <- function(par, X, Y) {
  X <- as.matrix(cbind(1, X))
  beta <- par[1:ncol(X)]
  sigma2 <- exp(par[ncol(X) + 1])
  -1/2*sum(log(sigma2) + ((Y - X %*% beta)^2)/sigma2)
  }
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  }
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- Calling it:

```
loglik(c(2,1,2,1,33,4,2),x,y)
loglik(c(2,1,2,1,33,4,7),x,y)
loglik(c(2,1,2,1,33,4,5),x,y)
```

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  - Expected number of Dem electoral college delegates
  - Probability that Dem candidate is elected: gets more than $\sum_{i=1}^{n} C_i/n > 0.5$ proportion of electoral college delegates

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     - Quiz: How might we also include estimation uncertainty?

Predictive Distribution of Electoral College Delegates

Including fundamental and estimation uncertainty

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       i.e. P(unknown|data). (Details shortly.)

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  - Calculate total Dem delegates nationally: add simulated winnings from all states: $\sum_{i=1}^{51} 1(\tilde{y}_{i,2020} > 0.5) C_i$
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How to draw simulations of $y_{i,2020}$

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$\rightsquigarrow$ We can now simulate the number of Democratic delegates, in repeated elections, with fundamental and estimation uncertainty represented

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Useful to connect to the literature. Feel free to ignore

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   - Or, in our preferred notation, draw $\tilde{y}_{i,2020}$ from $N(X_{i,2020}\tilde{\beta}, \tilde{\sigma}^2)$

Forecasting Errors for 1992 (forecasts from early October)

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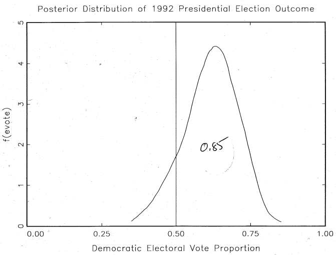
- Predictive distribution of electoral vote proportion:



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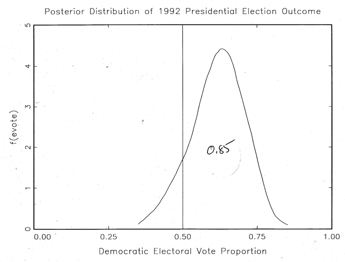

- Probability of Dem (Bill Clinton) victory:

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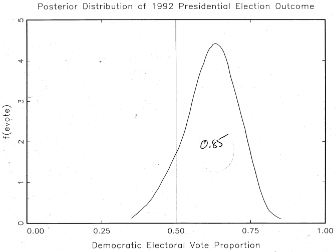

- Probability of Dem (Bill Clinton) victory: 0.85

Forecasting Errors for 1992 (forecasts from early October)

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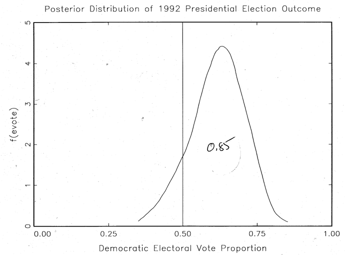

- Probability of Dem (Bill Clinton) victory: 0.85
- Error in Democratic 2-party electoral vote proportion:

Forecasting Errors for 1992 (forecasts from early October)

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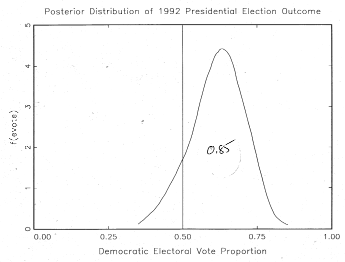

- Probability of Dem (Bill Clinton) victory: 0.85
- Error in Democratic 2-party electoral vote proportion: 0.01

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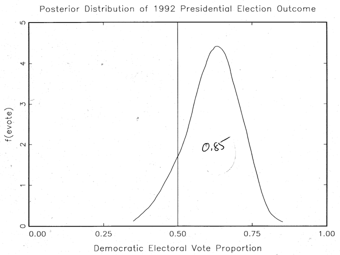

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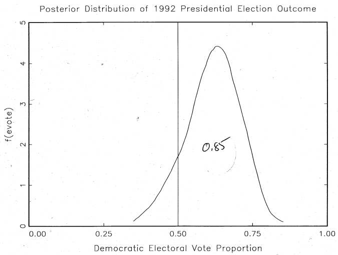

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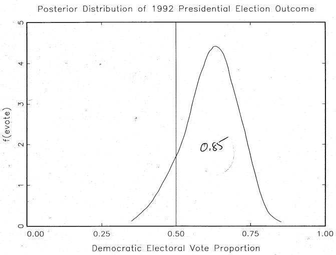

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- Error in Democratic 2-party electoral vote proportion: 0.01
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- Quiz: How big do you expect these errors will be if the model is correct and the election were run again?

The Impossibility of Inference Without Assumptions

Three Theories of Inference: Overview

Likelihood: Example, Derivation, Properties

Uncertainty in Likelihood Inference

Simulation from Likelihood Models

Extending the Linear Model with a Variance Function

A Gaussian Variance Function Model

A Gaussian Variance Function Model

The Model

A Gaussian Variance Function Model

The Model

1. $Y_i \sim N(y_i | \mu_i, \sigma_i^2)$

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A Gaussian Variance Function Model

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Any questions?